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SOME REMARKS ON AN ODE-SOLVER OF KIRCHGRABER

Sylvia Novo
Jesús Rojo

Departamento de Matemática Aplicada a la Ingeniería

E.T.S. de Ingenieros Industriales, Universidad de Valladolid

47011 Valladolid

Abstract

We suggest some modifications to a method of Kirchgraber which result in improved accuracy and shorter computation time.

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Introduction. The LIPS-code described by Kirchgraber in [3] is an integration procedure based on the method of averaging. It is intended for perturbed systems such that the unperturbed one has only periodic solutions with a common period T and the perturbation $\varepsilon f(t,x,\varepsilon)$ is also T-periodic. This method gives approximations to the real solution for values of the variable which are integer multiples of the period T, as long as nT remains in some interval $[0, L/|\varepsilon|]$ for a fixed L. The basic idea of the ODE-solver is to use the Poincaré-map to define a new system whose solutions are close to the real ones. The main characteristic of these new equations is that they can be integrated numerically with large step-size.

1. First we correct a minor error in [3]. In page 637 one reads

$$\dot{y} = q_{\bar{P}}(y)\,, \qquad y(0) = \xi \qquad (6)$$

whereas it should be

$$\dot{y} = \frac{1}{T} q_{\bar{P}}(y)\,, \qquad y(0) = \xi\,.$$

Consequently, expressions (8) and (9) in the same page should be also substituted by

$$\bar{\Phi}^{NO}_{\frac{1}{T} q_{\bar{P}}}(H, \xi)$$

and

$$\bar{\Phi}^{n}_{\frac{1}{T} q_{\bar{P}}}(H, \xi)\,, \qquad n = 1, \ldots, NO$$

respectively.

Alternatively, if (6) is kept unchanged, we may modify the expresion in (7) to

$$H = L/(T|\varepsilon|NO)$$

and then we use $\bar{\Phi}^{n}_{q_{\bar{P}}}(H, \xi)\,,\ n = 1, \ldots, NO$ to approximate $\Phi_f(nTH, \xi)$.

2. Kirchgraber proposes to use a Runge-Kutta method with constant stepsize for the calculation of the Poincaré-map and then solve numerically the equation

$$\dot{y} = \frac{1}{T} q_{\bar{P}}(y)\,. \qquad (1)$$

Our experience shows that the computation of the Poincaré-map with the help of a special method for perturbed systems reduces considerably the number of evaluations in Kirchgraber's method. A Runge-Kutta method will be still used to integrate (1). We will illustrate this fact with the example discussed by Kirchgraber in [3]. He integrates the single harmonic oscillator with a cubic nonlinearity

$$\ddot{y} + y = \varepsilon y^3\,, \qquad y(0) = 1\,, \quad \dot{y}(0) = 0\,. \qquad (2)$$

In [1], Bettis modified the classical difference methods of numerical integration to integrate certain products of an ordinary polynomial and a Fourier polynomial without truncation error. For instance, by modifying two coefficients of the method of Adams-Bashforth of $n+1$ steps he derived a method which integrates exactly the function

$$P_{n-2}(t) + a_0 \cos \omega t + b_0 \sin \omega t$$

where $P_{n-2}(t)$ is an ordinary polynomial of degree $n-2$ and ω a fixed frecuency.

We have applied this method (with three, five, seven and eight steps) to find the Poincaré-map of (2), obtaining

Table 1: LIPS-code of order 1 with 3-step Bettis.

ε	Δ	NI	NO	G
10^{-3}	$5.5 \cdot 10^{-3}$	40	2	$1.742 \cdot 10^3$
10^{-4}	$6.5 \cdot 10^{-4}$	90	3	$4.563 \cdot 10^3$
10^{-5}	$3.4 \cdot 10^{-5}$	170	3	$7.683 \cdot 10^3$

We have kept the format and the symbolism of the tables in [3] the better to visualize the efficiency of this modification. We may see that the number of function evaluations necessary to get the same precision obtained in [3] is now roughly divided by ten.

3. Our third observation concerns a class of equations to which Kirchgraber applies his method. These are the perturbed conservative oscillators, i.e., equations of the form

$$\ddot{y} + \omega^2 y = \varepsilon f(y)\,. \tag{3}$$

Table 2: LIPS-code of order 2 with 5-step Bettis.

ε	Δ	NI	NO	G
10^{-3}	$0.9 \cdot 10^{-5}$	80	4	$1.383 \cdot 10^{4}$
10^{-4}	$1.5 \cdot 10^{-7}$	250	6	$4.727 \cdot 10^{4}$
10^{-5}	$1.1 \cdot 10^{-9}$	470	12	$1.632 \cdot 10^{5}$

Table 3: LIPS-code of order 3 with 7-step Bettis.

ε	Δ	NI	NO	G
10^{-3}	$5.2 \cdot 10^{-8}$	110	8	$5.897 \cdot 10^{4}$
10^{-4}	$4.3 \cdot 10^{-11}$	200	15	$1.632 \cdot 10^{5}$
10^{-5}	$8.9 \cdot 10^{-14}$	500	35	$7.903 \cdot 10^{5}$

The key to the succes of Kirchgraber's method lies in the fact that it integrates T-periodic solutions exactly (not taking into account roundoff errors) and therefore makes good long-term predictions for solutions with periods close to the unperturbed period T.

We will show that it is possible to make a first approximation to the real period $T(\varepsilon)$ which results in a spectacular improvement in accuracy and computation time; this modification has been already applied to Bettis methods by Ferrándiz and Novo [2]. We remark that although it is theoretically possible to find the exact period $T(\varepsilon)$ by numerical or symbolic means, our procedure requires a much smaller additional computation. This modification will be referred to as the NEP-code (for Near Exact Period).

Table 4: LIPS-code of order 4 with 8-step Bettis.

ε	Δ	NI	NO	G
10^{-2}	$5.5 \cdot 10^{-7}$	170	6	$8.174 \cdot 10^{4}$
10^{-3}	$5.2 \cdot 10^{-11}$	250	20	$3.557 \cdot 10^{5}$
$2 \cdot 10^{-4}$	$8.5 \cdot 10^{-14}$	500	45	$1.385 \cdot 10^{6}$

For equation (3) we use the approximation

$$\frac{2\pi}{\sqrt{\omega^2 - a\varepsilon}}$$

where the value of a is calculated to eliminate secular first order terms. Usually, this is achieved by finding the coefficient of $\cos t$ in the cosine series of $f(\cos t)$. We have applied this modification of the period to the same example (2) obtaining the following results:

Table 5: NEP-code of order 1 with 8-step Bettis.

ε	Δ	NI	NO	G
10^{-3}	$2.9 \cdot 10^{-12}$	300	1	$5.096 \cdot 10^{3}$
10^{-4}	$3.8 \cdot 10^{-15}$	500	1	$7.696 \cdot 10^{3}$
10^{-5}	$5.8 \cdot 10^{-18}$	900	1	$1.290 \cdot 10^{4}$

In this particular case $T = 2\pi/\sqrt{1 - 0.75\varepsilon}$ and we have also taken $L = T$. According to table 5 the NEP-code (of order 1) for $\varepsilon = 10^{-3}, 10^{-4}, 10^{-5}$ using less than $2 \cdot 10^{4}$ functions evaluations, provides a result which is correct up

to an error

$$2.9 \cdot 10^{-12}, \quad 3.8 \cdot 10^{-15}, \quad 5.8 \cdot 10^{-18}$$

respectively. In contrast, the LIPS-code of order 1, using almost the same number of evaluations (see table 1 in [3]) yields a result which is correct up to an error

$$6.6 \cdot 10^{-3}, \quad 6.6 \cdot 10^{-4}, \quad 6.8 \cdot 10^{-5}$$

respectively. Thus we gain 9, 11 and 13 digits in each case.

We have performed the numerical experiments with quadruple precision (the double precision of NOS-VE FORTRAN) on a CYBER-930 machine.

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