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Singularities Arising in the Linearized Magnetohydrodynamics Equations

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ABSTRACT: The magnetohydrodynamics equations in a cylinder linearized around an equilibrium state with vertical magnetic field reduce to a single singular second order partial differential equation where the time frequency enters as a parameter. Although there could exist whole families of solutions depending on arbitrary values at the singularity [1], only one of them is physically correct. We analyze theoretically and numerically the problem and the physical consequences of our results.

AMS (MOS) subject classification: 76W05, 35C10

1 THE SETTING OF THE PROBLEM.

Let a stationary plasma fill a hollow cylinder $r_0 < r < r_1$, and assume that its density ρ and magnetic field $\mathbf{B} = (0, 0, B)$ depend only on the radius r, and that it satisfies the polytropic state equation $p = A\rho^\gamma$. For a fixed Fourier mode l in z, let u be the Fourier-Laplace transform $(t \to \omega)$ of a small perturbation p_* of the total (kinetic plus magnetic) pressure. If the values of the perturbed velocity and field are zero at $t = 0$, u satisfies

$$\operatorname{div}\left(\frac{1}{\omega^2\rho - l^2B^2}\nabla u\right) + \frac{1}{\gamma p}\,\frac{\omega^2\rho - l^2\gamma p}{\omega^2\rho\left(1+\dfrac{B^2}{\gamma p}\right) - l^2B^2}\,u = 0\,. \tag{1}$$

The points where the local Alfvén frequency $\omega_A = \pm\, l\, B/\sqrt{\rho}$ matches the time frequency ω are singular and very important in the study of magnetic oscillations, [2, 3, 4]. To simplify the study of this singularity, we will consider only the principal part of the differential operator and set the second term of (1) to zero; let $F = \omega^2\rho - l^2B^2$. After a Fourier transform $\theta \to n$ in the angle variable, (1) becomes

$$u'' + \left(\frac{1}{r} - \frac{F'(r)}{F(r)}\right)\, u' - \frac{n^2}{r^2}\, u = 0\,. \tag{2}$$

At any simple zero r_0 of F, F'/F behaves as $1/(r-r_0)$. For further notational simplification, and without affecting the properties of the equation we will take $l^2B^2 = 1$ and $\rho(r) = r$; for $\omega^2 = 1$, (2) becomes

$$u'' + \left(\frac{1}{r} - \frac{1}{r-1}\right) u' - \frac{n^2}{r^2}\, u = 0\,,$$

which we recognize as an hypergeometric equation. Its classical theory [5, 6] allows us, after a cumbersome but essentially straightforward calculation, to write two independent solutions in $B(1,1)$ as

$$v_1(r) = r^n(1-r)^2 \sum_{k=0}^{\infty} \frac{(\alpha(n))_k(\beta(n))_k}{(3)_k}\,\frac{(1-r)^k}{k!}$$

$$v_2(r) = -\,\frac{2\,r^n}{(\alpha(n)-2)_2\,(\beta(n)-2)_2} +$$

$$+\,r^n(1-r)\sum_{k=0}^{\infty}\frac{(\alpha(n))_k(\beta(n))_k}{(3)_k}\,(\Psi(\alpha(n)+k)+\Psi(\beta(n)+k)-$$

$$-\Psi(3+k)-\Psi(1+k))\,\frac{(1-r)^{k+2}}{k!} + \log(1-r)v_1(r)\,,$$

where $(x)_k = ,\,(x+k)/,\,(x)$, $\alpha(n) = n+3/2+\sqrt{n^2+1/4}$, $\beta(n) = n+3/2-\sqrt{n^2+1/4}$, and Ψ is the logarithmic derivative of the , function.

2 MAIN RESULTS.

We see that $v_2(r) = w_2(r) + \log(1-r)v_1(r)$, with v_1 and w_2 real functions. Also notice that the wronskian determinant $w_2'v_1 - w_2v_1'$ behaves as $a(r-1) + O\,(r-1)^2$ near 1, with $a = 4/(\alpha(n)-2)_2\,(\beta(n)-2)_2 > 0$. These function are $\mathcal{C}^1$ and, except for the term $\log(1-r)$, real valued in $[r_0, r_1]$. In the general case we would have $\log(r(\omega)-r)$, where $r(\omega)$ is the (complex) zero of F for ω near ± 1. Hence, for $\delta > 0$, $\log(1-(1+\delta)) = \log(1-(1-\delta)) + i\,\sigma\pi$, where $\sigma = \pm 1$, according to which branch of the logarithm is chosen. Let us study the correct determination of this imaginary part.

Lemma 1 *Let $\omega_0 = \pm 1$. Then σ is the sign of $\omega_0/F'(1)$ ($F'(1) = 1$ in our case).*

Proof. What we denote by u is in fact the Fourier-Laplace transform $\int_0^\infty e^{i\,\omega t}p_*(t,\mathbf{x})\,dt$; hence the solution of (1) for real ω should be the limit of (damped) solutions for $\omega + i\,\varepsilon$, $\varepsilon \downarrow 0$. Let $\omega(r)$ be the (complex) frequency which makes $F(\omega(r), r) = 0$ near $(\omega_0 = \pm 1,\ r = 1)$. Then

$$\frac{\partial F}{\partial \omega}\,\omega' + F' = 0\,,$$

that is, $2\omega(r)\rho(r)\omega'(r) + F'(r) = 0$. This implies

$$\omega(r) - \omega(1) = -\,\frac{F'(\omega(1),1)}{2\omega(1)\rho(1)}\,(r-1) + \mathrm{O}\,(r-1)^2\,;$$

thus, if $r(\omega)$ is the (complex) zero of $F(\omega, r(\omega))$ near ($\omega = \omega_0$, $r = 1$),

$$r(\omega) - r(\omega_0) = r(\omega) - 1 = -\frac{2\omega_0\rho(1)}{F'(\omega_0, 1)}\,(\omega - \omega_0) + \mathrm{O}\,(\omega - \omega_0)^2 .$$

Hence the sign of the imaginary part of $r(\omega_0 + i\,\varepsilon)$ is the sign of $-\omega_0/F'(1)$. When positive, $r(\omega)$ lies in the lower half plane and when r goes from $1-\delta$ to $1+\delta$, $r(\omega) - r$ goes from $r(\omega) - 1 + \delta$ to $r(\omega) - 1 - \delta$ in the lower half plane, so that its argument changes from 0 to $-\pi$; otherwise, from 0 to π. Always $\sigma = \mathrm{sgn}(\omega_0/F'(1))$. $\square$

How this fact relates to energy absorption? The increase of energy in X is given by

$$\frac{dE}{dt} = -\int_{\partial X} p_*\,\mathbf{v}\cdot\mathbf{n}\,d\sigma$$

(where $\mathbf{n}$ denotes the outer normal). It may be shown that the Fourier transforms P and V of p_* and $\mathbf{v}\cdot\mathbf{n}$ satisfy $V = (i\,\omega/F)\,P'$. Since we are dealing with a single resonant sheet, assume that p_* has only the Fourier modes $\pm l$ in z (so that $F = \omega^2\rho - l^2 B^2$ is fixed in the problem). We will study the integral above for a cylindrical surface $r = a$. As we know, under fairly general conditions, and with a somewhat relaxed notation,

$$\begin{aligned}
&\int_{\mathbb{T}\times\mathbb{R}} p_*(t,a,\theta,z)\,\mathbf{v}\cdot\mathbf{n}(t,a,\theta,z)\,d\theta\,dz = \\
&\quad= \int_{\mathbb{Z}\times\mathbb{R}} \widehat{p}_*(t,a,n,l)\;\widecheck{\mathbf{v}\cdot\mathbf{n}}\;(t,a,n,l)\,dn\,dl \\
&\quad= \sum_{n\in\mathbb{Z}} (\,(P(t,a,n,l)\,V(t,a,-n,-l) + P(t,a,n,-l)\,V(t,a,-n,l)\,)\,.
\end{aligned}$$

Hence the output of energy through $r = a$ is

$$-\sum_{n\in\mathbb{Z}} (\,(P(t,a,n,l)\,V(t,a,-n,-l) + P(t,a,n,-l)\,V(t,a,-n,l)\,)\,. \tag{3}$$

Assume also that the boundary conditions determine a one-dimensional space of solutions depending only on ω^2 (such as decay at infinite, $u(r_2 = \infty) = 0$, or a solid wall $u'(r_1) = 0$, etc.; most real problems are in this class). We may construct a generator of the space of solutions as follows:

Let $\alpha = \lambda_1 v_1 + \lambda_2 v_2$, λ_1, λ_2 real; $\beta = \delta\pi\lambda_2 v_1$, with $\delta = 0$ for $r < 1$ or $\delta = 1$ for $r \geq 1$. Then

$$\begin{aligned}
P &= \lambda(1,n,l)\,(\alpha + i\,\beta)\,e^{i\,t} + \lambda(-1,n,l)\,(\alpha - i\,\beta)\,e^{-i\,t}\,,\\
V &= \frac{i}{F}\,\lambda(1,n,l)\,(\alpha' + i\,\beta')\,e^{i\,t} - \frac{i}{F}\,\lambda(-1,n,l)\,(\alpha' - i\,\beta')\,e^{-i\,t}\,.
\end{aligned}$$

Since p_* is real, $P(-\omega,-n,-l) = \overline{P(\omega,n,l)}$, so the same happens for the coefficient λ. Also dE/dt is real, so it is enough to take the real part of the terms in (3). The real part of the non-oscillatory terms (that is, the sums in (3) not affected by $e^{2i\,t}$ or $e^{-2i\,t}$; the time mean of these is zero and hence they are not relevant in the net energy increase) is therefore

$$\frac{1}{F}\,(|\lambda(-1,n,l)|^2 + |\lambda(1,n,l)|^2)\,(\alpha'\beta - \alpha\beta')$$

for a fixed mode n. The wronskian determinant $\alpha'\beta - \alpha\beta'$ vanishes for $r < 1$, and satisfies $W' = (F'/F - 1/r)W$; hence $W = A\,F/r$ for $r \geq 1$, A a constant. One easily gets

$$A = \frac{r}{F}\,\lambda_2^2\pi(v_2'v_1 - v_2v_1') = \frac{r}{F}\,\lambda_2^2\pi(w_2'v_1 - w_2v_1' + \frac{1}{r-1}\,v_1^2)\,,$$

which as shown before, is positive at $r = 1$. (Recall that $F = r - 1$, $w_2'v_1 - w_2v_1' = a(r-1) + \mathrm{O}\,(r-1)^2$, $a > 0$, $v_1^2 = \mathrm{O}\,(r-1)^4$.)

Hence for a set X bounded for $r = a$ and $r = b$, dE/dt is a continuous function of (a, b) whenever both are less than or greater than 1; however, if $a < 1$, $b > 1$, dE/dt jumps by a fixed amount. We have proved

Theorem 1 *Energy absorption due to alfvenic resonance takes place at the singular set.*

Notice that the fact that $\beta \neq 0$ (that is, that an imaginary part occurs at the solution after the singularity $r = 1$) is essential for the existence of this phenomenon. Any solution not taking this term in consideration is physically erroneous. This would happen if one applies directly any of a number of numerical methods of integration which tolerate the singular point, but do not get the solution analytic in ω, as we will show next.

3 NUMERICAL RESULTS.

The integration of the initial value problem in a given interval (in our case $r \in [0.5, 1.5]$) is delicate because of the singularity ar $r = 1$. If however, one takes $r = x + i\,\varepsilon$, $x \in [0.5, 1.5]$, $\varepsilon > 0$, not only the singularity dissapears, but what we obtain is the correct solution in the limit $\varepsilon = 0$. This has been done in equation (2) for initial values $u(0.5) = 1$, $u'(0.5) = 1$, $n = 1, 2$, and $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$ (see figure 1). Behaviour for $n = 2$ is similar, as shown in figure 2. Observe that the

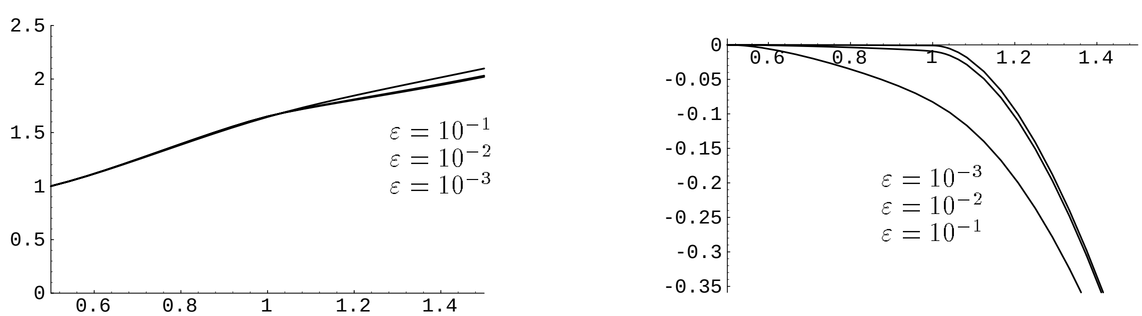

Figure 1: Left: real part of the solutions, for $n = 1$; right: imaginary part of the solutions, for $n = 1$.

behaviour of the imaginary part is as predicted: zero up to the singularity $r = 1$, and then a nontrivial function. By contrast, direct integration of (1) with $\varepsilon = 0$, even if the algorithm somewhat surpasses the critical point at $r = 1$, will never generate an

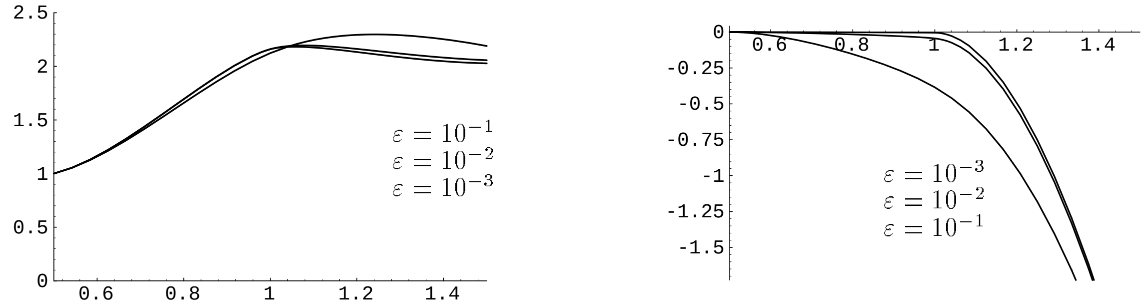

Figure 2: Left: real part of the solutions, for $n = 2$; right: imaginary part of the solutions, for $n = 2$.

imaginary part for a real equation with real initial value. We have checked that for $\varepsilon = 0$; the real part is correctly obtained but the imaginary part of u is always zero, as one would expect.

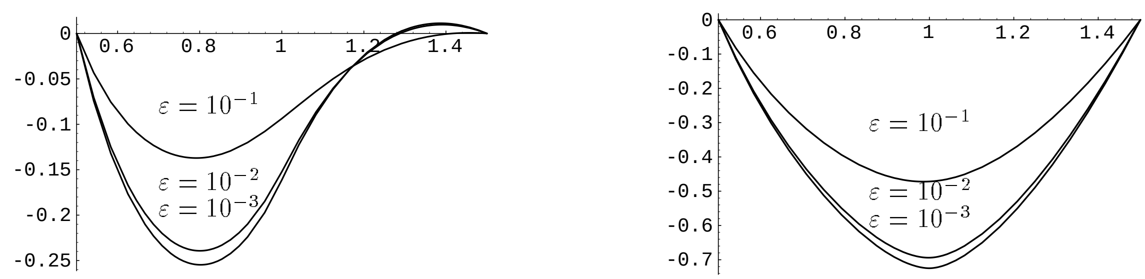

Figure 3: Boundary value problems for $n = 2$; real and imaginary parts of the solution.

As for the boundary problem, the same method may be used. The regular boundary value problems obtained by considering $r = x + i\,\varepsilon$, $x \in [0.5, 1.5]$, with the conditions $u(0.5) = 0 = u(1.5)$ and $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$, show the convergence of the solutions to the correct physical solution. For $n = 2$, we have the results of the figure 3.

These graphics have been obtained by using the shooting method with Mathematica, which allows a high degree of interactivity and is specially good at dealing with singularities. If we execute this method setting directly $\varepsilon = 0$, we still get an answer, but a completely wrong one (fig. 4).

In general, the behaviour of this integrator for $\varepsilon = 0$ is erratic. For $n = 1$, one gets the correct answer for some try values of $u'(0.5)$ and a wrong one for others. For $n = 2$ it is always false.

Finally, to emphasize the fact that one cannot choose the value of u at $r = 1$ arbitrarily and then solve the boundary problem at both intervals $[0.5, 1]$, $[1, 1.5]$, we have set $u(1) = 10$ and found the solutions for $n = 2$ for a slightly attenuated frequency $\omega = 1 + 0.1\,i$ (fig. 5). The solutions are obviously not differentiable. This

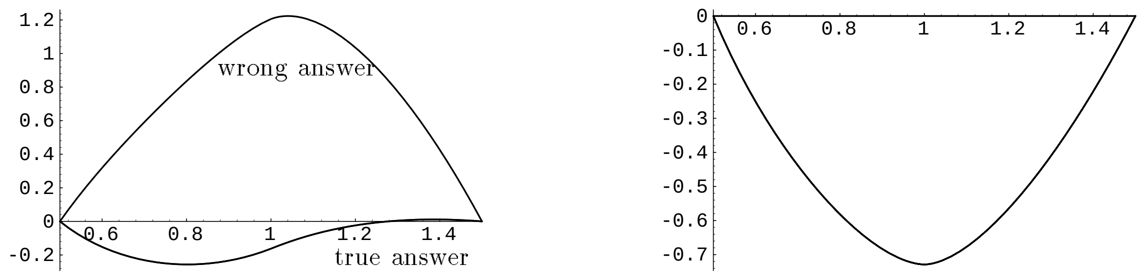

Figure 4: The wrong answer for $\varepsilon = 0$.

shows that the problem should not be treated as an equation with a singular boundary; one cannot impose arbitrary Dirichlet conditions at $r = 1$. Those are determined by the original problem in a complex variable setting.

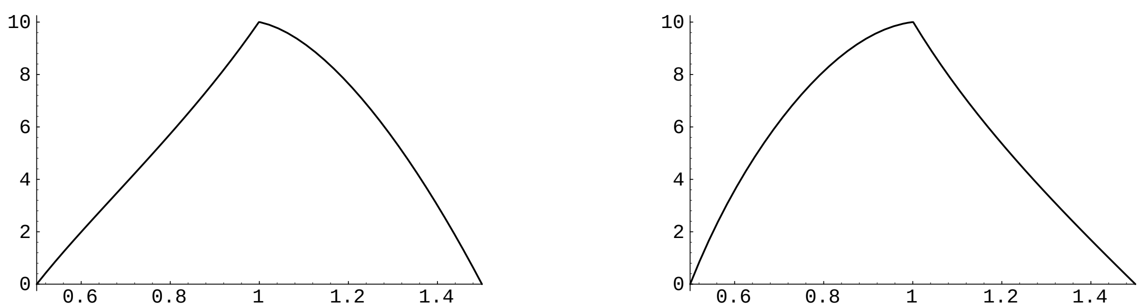

Figure 5: The solutions for $u(1) = 10$; real and imaginary part.

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REFERENCES

1. Núñez, M., & Tristán, L.A., A singular transmission problem: alfvenic resonance in Magnetohydrodynamics. (preprint, 1995)
2. Chen, L., & Hasegawa, A., Plasma heating by spatial resonance of Alfvén wave. Phys. of Fluids, Vol **17** (1974), pp. 1399-1403.
3. Chen, L., & Hasegawa, A.: Steady state excitation of field line resonance. J. of Geoph. Res., Vol **79** (1974), pp. 1024-1037.
4. Kivelson, M.G., & Southwood, D.J., Resonant ULF waves: a new interpretation. Geoph. Res. Lett., Vol **12** (1985), pp. 49-52.
5. Henrici, P., Applied and computational complex analysis. John Wiley and Sons, New York, 1976.
6. Whittaker, E.T., & Watson, G.N., A course of modern analysis. Cambridge Univ. Press, Cambridge, 1927.