(Preprint for)

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pp. 790-794 from Vol. 2 in
C. Perello, C. Simo and J. Sola-Morales(eds.)
EQUADIFF91, International Conference on Differential Equations, Barcelona, 1991
World-Scientific, Singapore, 1993

A LONG-TERM INTEGRATOR BASED ON KIRCHGRABER'S LIPS CODE

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Abstract

Kirchgraber derived in 1988 an integration procedure (called the LIPS-code) for long-term prediction of the solutions of equations which are perturbations of systems having only periodic solutions. In a previous paper we introduced some modifications of Kirchgraber's code which result in improved accuracy and shorter computation time. In this communication this new integrator is described and some examples are given to illustrate its performance.

1 Introduction.

The LIPS-code described by Kirchgraber in [3] is an integration procedure based on the method of averaging. It is intended for perturbed systems such that the unperturbed ones have only periodic solutions with a common period T, and the perturbation $\varepsilon\mathbf{f}(t,\mathbf{x},\varepsilon)$ is also T-periodic. This method approximates the real solution for values of the variable which are integer multiples of T, as long as nT remains in some interval $[0\,,\,L/|\varepsilon|]$ for a fixed L. The basic idea of the ODE-solver is to use the Poincaré-map to define a new system whose solutions are close to the real ones. The main characteristic of these new equations is that they can be integrated numerically with large step-size.

2 Description of the method.

First we describe briefly the LIPS-code. Let

$$\dot{\mathbf{x}} = \mathbf{f}(t,\mathbf{x},\varepsilon) := \mathbf{f}^0(\mathbf{x}) + \varepsilon\,\mathbf{f}^1(t,\mathbf{x},\varepsilon) \tag{1}$$

be a perturbed system, where ε is a so-called perturbation parameter, i.e. it is assumed that the range of values of ε is so small that the perturbation $\varepsilon\,\mathbf{f}^1(t,\mathbf{x},\varepsilon)$ is small compared to $\mathbf{f}^0(\mathbf{x})$. All solutions of the unperturbed system are periodic with a common period T and $\mathbf{f}^1$ is also T-periodic.

Given ξ and $L \geq 0$ the problem is to approximate the solution $\Phi_{\mathbf{f}}(t, \xi)$ of equation (1) corresponding to the initial value ξ taken at $t = 0$, for $t \in [0\,,\, L/|\varepsilon|]$, and for all ε suficiently small.

With the help of a one-step integration procedure to calculate the Poincaré-map a new system is defined

$$\dot{\mathbf{y}} = \frac{1}{T} q_{\bar{P}}(\mathbf{y}) \tag{2}$$

whose solutions are close to the solutions of (1). The last step is then to integrate numerically this equation with large step size. For more details about the calculation of $q_{\bar{P}}$ see [3].

In [4] we propose a double modification of Kirchgraber's code: first we take a first-order approximation of the perturbed period instead of the unperturbed one, and then we use a scheme specifically designed for integration of orbits to calculate $q_{\bar{P}}$. These modifications concern particularly a class of equations to which Kirchgraber applies his method. These are the perturbed conservative oscillators, i.e. equations of the form

$$\ddot{y} + \omega^2 y = \varepsilon\, f(y)\,. \tag{3}$$

The key to the success of Kirchgraber's method lies in the fact that it integrates T-periodic solutions exactly (not taking into account roundoff errors) and therefore makes good long-term predictions for solutions with period close to the unperturbed period T. We have shown in [4] that it is possible to make a first aproximation to the real period $T(\varepsilon)$ which results in a spectacular improvement in accuracy and computation time; this modification had been already applied to Bettis methods by Ferrándiz and Novo [2].

For equation (3) we use the approximation

$$\frac{2\pi}{\sqrt{\omega^2 - a\varepsilon}}$$

where the value of a is calculated to eliminate secular first order terms. Usually, this is achieved by finding the coefficient of $\cos t$ in the cosine series of $f(\cos t)$.

For the calculation of $q_{\bar{P}}$ for equation (3) we use a Bettis method which integrates exactly $\cos \omega t$ and $\sin \omega t$, (see [1]).

This double modification of the LIPS-code will be referred to as the NEP-code (for near exact period).

3 Numerical Experiments.

In this section we describe the application of the NEP-code to two systems of ODE's: an easy case of Duffing's equation and the two-body problem with perturbation J_2. The results are compared with those obtained by the application of the LIPS-code.

3.1 Duffing's equation.

We consider the single harmonic oscillator with a cubic nonlinearity

$$\ddot{y} + y = \varepsilon\, y^3\,, \qquad y(0) = 1 \quad \dot{y}(0) = 0\,. \tag{4}$$

We have applied the NEP-code of order 1 with 8-step Bettis obtaining the following results:

Table 1: NEP-code of order 1 with 8-step Bettis.

ε	Δ	NI	NO	G
10^{-3}	$2.9 \cdot 10^{-12}$	300	1	$5.096 \cdot 10^3$
10^{-4}	$3.8 \cdot 10^{-15}$	500	1	$7.696 \cdot 10^3$
10^{-5}	$5.8 \cdot 10^{-18}$	900	1	$1.290 \cdot 10^4$

We have kept the format and the symbolism of the tables in [3] the better to visualize the efficiency of this modification. Therefore $t_f := L/|\varepsilon|$, Δ is the error at the final point t_f, G is the total number of function evaluations and NI, NO are integer number such that $h = T/NI$ is the step used in the integration of the Poincaré-map and $H = L\,/(|\varepsilon|\, NO)$ is the step used in the integration of equation (2). In this case we have taken $L = T = 2\pi/\sqrt{1 - 0.75\,\varepsilon}$.

The application of LIPS-code of order 1 to the same problem gives the following table (see [3]):

Table 2: LIPS-code of order 1.

ε	Δ	NI	NO	G
10^{-3}	$6.6 \cdot 10^{-3}$	50	5	$4.225 \cdot 10^4$
10^{-4}	$6.6 \cdot 10^{-4}$	50	5	$4.225 \cdot 10^4$
10^{-5}	$6.8 \cdot 10^{-5}$	50	5	$4.225 \cdot 10^4$

with $L = T = 2\pi$.

Therefore, according to table 1 the NEP-code (of order 1) for $\varepsilon = 10^{-3}$, 10^{-4}, 10^{-5} using less than $2 \cdot 10^4$ function evaluations, provides a result which is correct up to an error

$$2.9 \cdot 10^{-12}\,, \quad 3.8 \cdot 10^{-15}\,, \quad 5.8 \cdot 10^{-18}$$

respectively. In contrast, the LIPS-code of order 1, using almost the same number of evaluations (see table 2) yields a result which is correct up to an error

$$6.6 \cdot 10^{-3}\,, \quad 6.6 \cdot 10^{-4}\,, \quad 6.8 \cdot 10^{-5}$$

respectively. Hence we gain 9, 11 and 13 digits in each case.

3.2 Integration of highly eccentric orbits.

We consider an equatorial satellite affected by oblateness perturbation as the basic problem. A highly eccentric orbit (eccentricity $e = 0.99$) is chosen to clarify the good performance of the procedure. We assume that the perigee is at an altitude of about 0.05 Earth radii and we consider only the perturbation due to the J_2-term. In this case, using focal variables (see [5] for more details), the equations of motion are reduced to

$$\left\{ \begin{array}{lcl} x_i'' & = & -x_i \qquad\qquad i = 1,2,3 \\ u'' & = & -u + \dfrac{1}{c^2} + \dfrac{3}{2} J_2\, u^2 \\ c' & = & 0 \\ t' & = & \dfrac{1}{cu^2} \end{array} \right. \tag{5}$$

where (x_1, x_2, x_3) are the three components of the direction vector of the particle, u is the inverse of the radial distance, c is the angular momentum and t the time. In this form, the integration of the equation for t' in (5) gives rise to great errors in the determination of the physical time, so it should be calculated by an alternative method. A good choice would be to use a time element (see [5] for instance). In this communication we have only taken into account the equations for x_i, $i = 1, 2, 3$ and u.

This problem has two different fundamental oscillation frequencies. The one for the first three equations is 1, and the approximation that we use for the frequency corresponding to

$$u'' = -u + \frac{1}{c^2} + \frac{3}{2} J_2\, u^2 \tag{6}$$

is $\sqrt{1 - 3J_2 B}$, where B is a constant such that the change of variable $u = v + B$ transforms (6) into

$$v'' + (1 - 3\, J_2\, B)v = \frac{3}{2} J_2\, v^2\,. \tag{7}$$

We have integrated separately each equation with their adequate frequency using the NEP-code of orders 1 and 2, obtaining the following results:

Table 3: NEP-code with 8-step Bettis.

order	Δ	NI	NO	G
1	$4.3 \cdot 10^{-10}$	200	1	$8.268 \cdot 10^3$
2	$2.9 \cdot 10^{-14}$	300	1	$2.106 \cdot 10^4$

In this case Δ is the relative error in cartesian coordinates after one thousand revolutions from the perigee (that is, more than 170 years).

With the application of the LIPS-code to the same problem we obtain:

Table 4: Lips-code.

order	Δ	NI	NO	G
1	$2.3 \cdot 10^{-3}$	50	5	$4.225 \cdot 10^{4}$
2	$3.6 \cdot 10^{-5}$	100	10	$3.380 \cdot 10^{5}$
4	$4.6 \cdot 10^{-10}$	200	20	$2.704 \cdot 10^{6}$

Therefore, according to table 3 the NEP-code (of order 1) using less than 10^4 function evaluations, provides a result which is correct up to an error $4.3 \cdot 10^{-10}$, while with the same number of evaluations, the error in the LIPS-code (table 4) is $2.3 \cdot 10^{-3}$. To get an error in trayectory of 0.5 cm ($\simeq 5 \cdot 10^{-10}$ relative error) with the LIPS-code, we should use the fourth order one and then more than 10^6 function evaluations are needed. Moreover, we can improve the precision by use the NEP-code of order 2 which yields a result which is correct up to an error $2.9 \cdot 10^{-14}$, using less than $3 \cdot 10^4$ function evaluations.

We have performed the numerical experiments with quadruple precision (the double precision of NOS-VE FORTRAN) on a CYBER-930 machine.

Acknowledgement. The authors thank the financial support received from CICYT of Spain under Project EPS.88-0541.

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