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A CLASS OF ERGODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENTS

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ABSTRACT: The existence of a class of ergodic solutions, pseudo almost periodic ($\mathcal{PAP}$) and generalized pseudo almost periodic ($\widetilde{\mathcal{P}}\mathcal{AP}$) solutions, of differential equations with piecewise constant arguments is investigated by introducing a new tool, a class of ergodic sequences.
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1. INTRODUCTION

The existence of ergodic solutions, which are bounded solutions with mean values, of differential equations has received much attention because of the obvious practical importance of it in many fields of applied sciences. Recently, a new class of ergodic functions, pseudo almost periodic ($\mathcal{PAP}$) functions, was introduced and applied to ordinary and partial differential equations in [**3-5**],[**15**]. Up to now, there have not been any papers concerning the existence of $\mathcal{PAP}$ solutions of differential equations with piecewise constant arguments (pca) which were considered by K.L.Cooke, J.Wiener etc. and found application in some physical and biomedical problems (see [**1**], [**2**], [**6-9**],[**11-14**]). In this paper we study the existence of pseudo almost periodic solutions and generalized pseudo almost periodic solutions for the following differential equations with pca

$$x'(t) = a\,x(t) + \sum_{i=-N}^{N} a_i\, x([t+i]) + f(t), \qquad N \geq 2, \tag{1}$$

$$x'(t) = a\,x(t) + \sum_{i=-N}^{N} a_i\, x([t+i]) + g(t, x(t), x([t])), \qquad N \geq 2, \tag{2}$$

where $[\cdot]$ denotes the greatest integer function, a, a_i are constants, $f \in \widetilde{\mathcal{P}}\mathcal{AP}(R,R)$ ($\mathcal{PAP}(R,R)$), $g \in \widetilde{\mathcal{P}}\mathcal{AP}(R\times R^2, R)$ ($\mathcal{PAP}(R\times R^2,R)$) is bounded, and there exists a constant $\eta > 0$ such that

$$|g(t,x_1,y_1) - g(t,x_2,y_2)| \leq \eta(|x_1 - x_2| + |y_1 - y_2|), \text{ for every } (t,x_1,y_1), (t,x_2,y_2) \in R\times R^2. \tag{3}$$

In the rest of this section the required definitions and our main results are given. A class of ergodic sequences, which are the main tool in the proofs of our main results, is introduced and discussed in Section 2. Proofs of our results are given in Section 3.

Let $C(R,R^n)$ (respectively $C(R\times\Omega, R^n)$, where $\Omega \subset R^{2n}$) denote the Banach space of bounded continuous functions $\varphi(t)$ (res. $\varphi(t,x)$) from R (res. $R\times\Omega$) to R^n with norm $||\varphi|| = \sup_{t\in R}|\varphi(t)|$ (res. $||\varphi|| = \sup_{t\in R, x\in\Omega}|\varphi(t,x)|$), where $|\cdot|$ is the Euclidean norm.

Definition 1. *A function $f \in C(R,R^n)(C(R\times\Omega,R^n))$ is called pseudo almost periodic (see* [**3** $-$ **5**], [**15**]) *if $f = f_1 + f_0$, where f_1 is almost periodic in $t\in R$ (almost periodic in $t \in R$, uniformly in $x\in\Omega$), and $f_0 \in \mathcal{PAP}_0(R,R^n)$ ($\mathcal{PAP}_0(R\times\Omega,R^n)$), where*

$$\mathcal{PAP}_0(R,R^n) = \{\varphi \in C(R,R^n):\ m(|\varphi|) = \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}|\varphi(t)|dt = 0\},$$

$$\mathcal{PAP}_0(R\times\Omega,R^n)=\{\varphi\in C(R\times\Omega,R^n):\ m(|\varphi|)=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}|\varphi(t,x)|dt=0 \textit{ uniformly in } x\in\Omega\}.$$

$\mathcal{PAP}_0(R,R^n)$ is a translation invariant closed ideal of $C(R,R^n)$. f_1 and f_0 are called the almost periodic component and the ergodic perturbation, respectively, of the function f. Denote by $\mathcal{PAP}(R,R^n)$ ($\mathcal{PAP}(R\times\Omega,R^n)$) the set of all such functions f.

Definition 2. *A Lebesgue measurable function f from R to R^n (res. from $R\times\Omega$ to R^n) is called generalized pseudo almost periodic (see [**3, 4**]) if $f=f_1+f_0$, where f_1 is almost periodic in $t\in R$ (almost periodic in $t\in R$, uniformly in $x\in\Omega$), and $f_0\in\widetilde{\mathcal{P}}\mathcal{AP}_0(R,R^n)$ ($\widetilde{\mathcal{P}}\mathcal{AP}_0(R\times\Omega,R^n)$) where*

$$\widetilde{\mathcal{P}}\mathcal{AP}_0(R,R^n)=\{\varphi:R\to R^n \textit{ Lebesgue measurable, such that } m(|\varphi|)=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}|\varphi(t)|dt=0\},$$

$$\widetilde{\mathcal{P}}\mathcal{AP}_0(R\times\Omega,R^n)=\left\{\begin{array}{l}\varphi:R\times\Omega\to R^n,\ \textit{such that } \varphi(\cdot,x)\in\widetilde{\mathcal{P}}\mathcal{AP}_0(R,R^n), \textit{for each } x\in\Omega \textit{ and}\\ \qquad m(|\varphi|)=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}|\varphi(t,x)|dt=0 \textit{ uniformly in } x\in\Omega\end{array}\right\}.$$

f_1 and f_0 are called the almost periodic component and the ergodic perturbation, respectively, of the function f. Denote by $\widetilde{\mathcal{P}}\mathcal{AP}(R,R^n)$ ($\widetilde{\mathcal{P}}\mathcal{AP}(R\times\Omega,R^n)$ the set of all such functions f.

Definition 3. *A function $x:R\to R$ is called a solution of Eq.(1) (or(2)) if the following conditions are satisfied*

(1) *x is continuous on R,*
(2) *the derivative $x'(t)$ of $x(t)$ exists everywhere, with possible exception of the point $[t]$, where one-sided derivatives exist,*
(3) *x satisfies Eq.(1) (or(2)) on each interval $[n,n+1]$, $n\in Z=\{\cdots,-1,0,1,\cdots\}$(see [**1**], [**2**], [**6 − 9**], [**11 − 14**]).*

Obviously, if $x(t)$ is a solution of Eq.(1) on R, then

$$x(t)=e^{a(t-n)}c_n+(e^{a(t-n)}-1)\sum_{i=-N}^{N}a^{-1}a_i\,c_{n+i}+\int_n^t e^{a(t-s)}f(s)ds,\qquad n\le t<n+1,\tag{4}$$

where $x(n+i)=c_{n+i}$, $-N\le i\le N$ (the discussion is the same for the case $a=0$). By using the continuity of a solution at any point, we get the following difference equation

$$c_{n+1}=e^a c_n+\sum_{i=-N}^{N}(e^a-1)a^{-1}a_i\,c_{n+i}+\int_n^{n+1}e^{a(n+1-s)}f(s)ds,\qquad n\in Z.\tag{5}$$

Let

$$\begin{aligned}&b_0=e^a+a^{-1}a_0(e^a-1),\qquad b_1=a^{-1}a_1(e^a-1)-1,\\&b_i=a^{-1}a_i(e^a-1),\qquad i=-1,\pm2,\cdots,\pm N,\\&h_n=-\int_n^{n+1}e^{a(n+1-s)}f(s)ds.\end{aligned}$$

Then (5) becomes

$$\sum_{i=-N}^{N}b_i\,c_{n+i}=h_n,\tag{6}$$

The corresponding homogeneous equation to Eq.(6) is

$$\sum_{i=-N}^{N}b_i\,c_{n+i}=0,\tag{7}$$

According to [**8**], we can get the particular solutions as $c_n = \lambda^n$ for the homogeneous difference equation (7). At this time, λ will satisfy the following equation

$$\sum_{i=-N}^{N} b_i\, \lambda^{n+i} = 0. \tag{8}$$

Our main results are as follows

Theorem 1. *Suppose that all roots of Eq.*(8) *are simple* (*denoted by* $\lambda_1, \cdots, \lambda_{2N}$) *and* $|\lambda_i| \neq 1,\ 1 \leq i \leq 2N$. *Then*

(1) *for any* $f \in \widetilde{\mathcal{P}}\mathcal{AP}_0(R,R)$ ($\mathcal{PAP}_0(R,R)$ *or* $\widetilde{\mathcal{P}}\mathcal{AP}_0(R,R) \cap M_b(R,R)$), *Eq.*(1) *has a solution* $x \in \widetilde{\mathcal{P}}\mathcal{AP}_0(R,R)$ ($\mathcal{PAP}_0(R,R)$), *furthermore, x is unique if* $f \in \mathcal{PAP}_0(R,R)$ *or* $f \in \widetilde{\mathcal{P}}\mathcal{AP}_0(R,R) \cap M_b(R,R)$, *where*

$$M_b(R,R) = \{\varphi :\ R \to R \quad \text{is Lebesgue measurable and bounded on } R\ \};$$

(2) *for any* $f \in \widetilde{\mathcal{P}}\mathcal{AP}(R,R)$ ($\mathcal{PAP}(R,R)$ *or* $\widetilde{\mathcal{P}}\mathcal{AP}(R,R) \cap M_b(R,R)$), *Eq.*(1) *has a solution* $x \in \widetilde{\mathcal{P}}\mathcal{AP}(R,R)$ ($\mathcal{PAP}(R,R)$), *and x is unique if* $f \in \mathcal{PAP}(R,R)$ *or* $f \in \widetilde{\mathcal{P}}\mathcal{AP}(R,R) \cap M_b(R,R)$.

Theorem 2. *Suppose that all roots of Eq.*(8) *are simple* (*denoted by* $\lambda_1, \cdots, \lambda_{2N}$) *and* $|\lambda_i| \neq 1,\ 1 \leq i \leq 2N$. *Then there exists* $\eta_* > 0$, *such that*

(1) *when* $0 \leq \eta < \eta_*$, *Eq.*(2) *has a unique solution* $x \in \mathcal{PAP}_0(R,R)$ *if* $g \in \widetilde{\mathcal{P}}\mathcal{AP}_0(R \times R^2, R)$ *is bounded and satisfies the Lipschitz condition* (3);

(2) *when* $0 \leq \eta < \eta_*$, *Eq.*(2) *has a unique solution* $x \in \mathcal{PAP}(R,R)$ *if* $g \in \widetilde{\mathcal{P}}\mathcal{AP}(R \times R^2, R)$ *is bounded and satisfies the Lipschitz condition* (3).

2. A CLASS OF ERGODIC SEQUENCES

To prove our main results, we introduce and discuss a class of ergodic sequences.

Definition 4.

(1) *A sequence* $x : Z \to R$ *is said to be a* $\widetilde{\mathcal{P}}\mathcal{AP}_0$ ($\mathcal{PAP}_0$) *sequence if it satisfies*

$$\lim_{n\to+\infty} \frac{1}{2n} \sum_{k=-n}^{n} |x(k)| = 0 \quad \Big(\lim_{n\to+\infty} \frac{1}{2n} \sum_{k=-n}^{n} |x(k)| = 0 \text{ and it is bounded}\Big).$$

(2) *A sequence* $x : Z \to R$ *is said to be a* $\widetilde{\mathcal{P}}\mathcal{AP}$ ($\mathcal{PAP}$) *sequence if* $x = x_1 + x_0$, *where* x_1 *is an almost periodic sequence* (*see* [**10, 14**]), *and* x_0 *is a* $\widetilde{\mathcal{P}}\mathcal{AP}_0$ ($\mathcal{PAP}_0$) *sequence.*

Remark 1.

(1) *A sequence vanishing at infinity is a* $\mathcal{PAP}_0$ *sequence. The* $\mathcal{PAP}_0$ *sequence* $x(n) = 1$ *if* $n = 2^k$ *and* $x(n) = 0$ *otherwise, shows that a* $\mathcal{PAP}_0$ *sequence is, in general, not a sequence vanishing at infinity.*

(2) *The sequence* $x(n) = |k|$ *if* $n = 2^{k^2}$ *and* $x(n) = 0$ *otherwise, is an example of an unbounded* $\widetilde{\mathcal{P}}\mathcal{AP}_0$ *sequence. In fact,*

$$\lim_{n\to+\infty} \frac{1}{2n} \sum_{k=-n}^{n} |x(k)| \leq \lim_{k\to+\infty} \frac{k(k+1)}{2^{k^2}} \leq \lim_{k\to+\infty} \frac{k+1}{k(k^2+1)} = 0.$$

Proposition 1. *Suppose that* $\{x(n)\}_{n\in Z}$ *is a* $\widetilde{\mathcal{P}}\mathcal{AP}_0$ $(\mathcal{PAP}_0)$ *sequence. Then there exists a function* $f \in \widetilde{\mathcal{P}}\mathcal{AP}_0(R,R)$ $(\mathcal{PAP}_0(R,R))$ *such that* $f(n) = x(n),\ n \in Z$.

Proof. Let x be a $\widetilde{\mathcal{P}}\mathcal{AP}_0$ sequence. And we set the function $f(t) = x(n),\ t \in [n, n+1),\ n \in Z$. Then f is a simple function on R, thus it is Lebesgue measurable. For every $T > 0$ there exists a $n \in Z$ such that $T \in [n, n+1)$. From

$$\frac{1}{2T}\int_{-T}^{T} |f(t)|dt \le \left(\frac{n+1}{n}\right)\left(\frac{1}{2(n+1)}\sum_{k=-(n+1)}^{n+1} |x(k)|\right) \longrightarrow 0 \quad \text{as } n \to +\infty,$$

it follows that $f \in \widetilde{\mathcal{P}}\mathcal{AP}_0(R,R)$.

Now let x be a $\mathcal{PAP}_0$ sequence. Then there exists a positive constant M such that $|x(n)| \le M$ for every $n \in Z$. We take the function $f(t) = (x(n+1) - x(n))(t-n) + x(n)$, $t \in [n, n+1)$, $n \in Z$, then f is continuous and $|f(t)| \le M$ for every $t \in R$, and for every $k \in Z$,

$$\int_{k}^{k+1} |f(t)|dt \le \frac{1}{2}\,|x(k+1)| + \frac{3}{2}\,|x(k)|.$$

For $T > 0$ and $T \in [n, n+1)$, $n \in Z^+$ we have

$$\frac{1}{2T}\int_{-T}^{T} |f(t)|dt \le \left(\frac{n+1}{n}\right)\left(\frac{1}{(n+1)}\sum_{k=-(n+1)}^{n+1} |x(k)|\right) \longrightarrow 0 \quad \text{as } n \to +\infty,$$

that is $f \in \mathcal{PAP}_0$. This completes the proof.

Remark 2. *The converse proposition of the above proposition is not true. We consider the function*

$$f(t) = \begin{cases} 2^{|k|}(t-k) + 1, & t \in [k - 2^{-|k|}, k), \\ -2^{|k|}(t-k) + 1, & t \in [k, k + 2^{-|k|}], \\ 0, & \text{otherwise}, \end{cases}$$

where $k \ne 0$. *Clearly,* $f \in C(R,R)$ *and* $\int_{-\infty}^{+\infty} |f(t)|dt = \int_{-\infty}^{+\infty} f(t)dt = 2$, *thus* $m(|f|) = 0$, *that is* $f \in \mathcal{PAP}_0(R,R)$. *But* $f(k) = 1$, *for every* $k \in Z - \{0\}$. *It follows that the sequence* $\{f(k)\}$ *is not a* $\mathcal{PAP}_0$ *sequence.*

Combining Proposition 1 with the results in A.M.Fink's book [**10**], it is deduced

Proposition 2. *If* $\{x(n)\}_{n\in Z}$ *is a* $\widetilde{\mathcal{P}}\mathcal{AP}$ $(\mathcal{PAP})$ *sequence, then there exists* $f \in \widetilde{\mathcal{P}}\mathcal{AP}(R,R)$ $(\mathcal{PAP}(R,R))$ *such that* $f(n) = x(n),\ n \in Z$.

Lemma 1. *If* $f \in \widetilde{\mathcal{P}}\mathcal{AP}_0(R,R)$ $(\mathcal{PAP}_0(R,R)$ *or* $\widetilde{\mathcal{P}}\mathcal{AP}_0(R,R) \cap M_b(R,R))$, *then the sequence*

$$h = \{h_n\}_{n\in Z} = \left\{\int_{n}^{n+1} e^{a(n+1-s)} f(s)ds\right\}_{n\in Z}$$

is a $\widetilde{\mathcal{P}}\mathcal{AP}_0$ $(\mathcal{PAP}_0)$ *sequence.*

Proof. Since

$$\frac{1}{2n}\sum_{k=-n}^{n} |h_k| \le \frac{1}{2n}\int_{-n}^{n+1} e^{|a|}|f(s)|ds \le \left(\frac{n+1}{n}\right)\frac{e^{|a|}}{2(n+1)}\int_{-(n+1)}^{n+1} |f(s)|ds$$

and $f \in \widetilde{\mathcal{P}}\mathcal{AP}_0(R,R)$ $(\mathcal{PAP}_0(R,R))$, it follows that h is a $\widetilde{\mathcal{P}}\mathcal{AP}_0$ $(\mathcal{PAP}_0)$ sequence.

3. PROOFS OF MAIN RESULTS

Proof of Theorem 1.

(1) Let $f \in \widetilde{\mathcal{P}}\mathcal{AP}_0(R,R)$ ($\mathcal{PAP}_0(R,R)$ or $\widetilde{\mathcal{P}}\mathcal{AP}_0(R,R) \cap M_b(R,R)$). Since all roots of Eq.(8) are simple,

$$\{k_1\lambda_1^n + \cdots + k_{2N}\lambda_{2N}^n\}$$

is the general solution of equation (7), where $k_1, \cdots, k_{2N}$ are any constants. Let $L = \{l \,/\, |\lambda_l| < 1,\, 1 \le l \le 2N\}$, $L' = \{l \,/\, |\lambda_l| > 1,\, 1 \le l \le 2N\}$. Then $L \cap L' = \emptyset$, $L \cup L' = \{1, \cdots, 2N\}$.

We define a sequence $\{c_n\}$ by

$$c_n = \sum_{l \in L} k_l \sum_{m \le n-1} \lambda_l^{n-(m+1)} h_m + \sum_{l \in L'} k_l \sum_{m \ge n} \lambda_l^{n-(m+1)} h_m, \tag{9}$$

where the unknown constants $k_l,\ 1 \le l \le 2N$, are determined later.

First of all, we show that the sequence $\{c_n\}$ is well-defined. In fact, for each fixed n, let

$$\begin{aligned} S_l(n) &= \sum_{m \le n-1} |\lambda_l|^{n-(m+1)} |h_m|, \qquad l \in L, \\ S_l'(n) &= \sum_{m \ge n} |\lambda_l|^{n-(m+1)} |h_m|, \qquad l \in L', \\ S_l(n,q) &= \sum_{q \le m \le n-1} |\lambda_l|^{n-(m+1)} |h_m|, \qquad l \in L, \\ S_l'(n,q) &= \sum_{n \le m \le q} |\lambda_l|^{n-(m+1)} |h_m|, \qquad l \in L'. \end{aligned}$$

Without loss of generality, we assume that $n-1 > 0$, and define

$$\varepsilon_n(q) = \frac{1}{n-q} \sum_{k=q}^{n-1} |h_k| \qquad \text{for } q \le n-1.$$

Since $\{h_n\}$ is a $\widetilde{\mathcal{P}}\mathcal{AP}_0$ sequence,

$$\lim_{q \to -\infty} \frac{1}{n-q} \sum_{k=q}^{n-1} |h_k| = \lim_{q \to -\infty} \frac{q}{n-q} \frac{1}{q} \sum_{k=q}^{0} |h_k| + \lim_{q \to -\infty} \frac{1}{n-q} \sum_{k=1}^{n-1} |h_k| = 0.$$

Thus for each fixed n, the sequence $\{\varepsilon_n(q) : \ q \in \{n-1, n-2, \cdots\}\}$ is a bounded sequence, that is, there exists a positive constant $M(n)$ such that $\varepsilon_n(q) \le M(n)$ for $q \in \{n-1, n-2, n-3, \cdots\}$. From

$$\begin{aligned} S_l(n,q) &= |\lambda_l|^{n-(q+1)}|h_q| + |\lambda_l|^{n-(q+2)}|h_{q+1}| + \cdots + |\lambda_l||h_{n-2}| + |h_{n-1}| \\ &= |\lambda_l|^{n-(q+1)}(n-q)\varepsilon_n(q) + (|\lambda_l|^{n-(q+2)} - |\lambda_l|^{n-(q+1)})(n-(q+1))\varepsilon_n(q+1) \\ &\quad + \cdots + (1 - |\lambda_l|)\varepsilon_n(n-1) \\ &\le M(n)(1 + |\lambda_l| + |\lambda_l|^2 + \cdots + |\lambda_l|^{n-(q+1)}) \\ &\le \frac{M(n)}{1 - |\lambda_l|} < +\infty, \end{aligned}$$

we know that $\sup_{q \le n-1} S_l(n,q) < +\infty$ for each fixed n, that is, $S_l(n)$ is well-defined. Similarly, we have $\sup_{q \ge n} S_l'(n,q) < +\infty$ for each fixed n, i.e., $S_l'(n)$ is also well-defined. Therefore, the sequence $\{c_n\}$ is well-defined on Z.

Now we prove the part (1) in Theorem 1.

Firstly, we choose $\{k_l\}$ such that the sequence $\{c_n\}$ defined by (9) is a solution of the difference equation (5).

Without loss of generality, we assume $L = \{l \,/\, 1 \le l \le l_0\}$, $L' = \{l \,/\, l_0+1 \le l \le 2N\}$. From (9) and (6), we obtain

$$\begin{aligned}\sum_{i=-N}^{N} b_i c_{n+i} &= \sum_{i=-N}^{N} b_i \sum_{l=0}^{l_0} k_l \sum_{m\le n+i-1} \lambda_l^{n+i-(m+1)} h_m \\ &\quad + \sum_{i=-N}^{N} b_i \sum_{l=l_0+1}^{2N} k_l \sum_{m\ge n+i} \lambda_l^{n+i-(m+1)} h_m \\ &= h_n.\end{aligned}$$

By using (8) and comparing the coefficients of h_n's, we get (also see [**14**]),

$$\begin{cases} \displaystyle\sum_{l=1}^{l_0} k_l - \sum_{l=l_0+1}^{2N} k_l = 0, \\ \cdots\cdots\cdots \\ \displaystyle\sum_{l=1}^{l_0} k_l\lambda_l^{N-2} - \sum_{l=l_0+1}^{2N} k_l\lambda_l^{N-2} = 0, \\ \displaystyle\sum_{l=1}^{l_0} k_l\lambda_l^{N-1} - \sum_{l=l_0+1}^{2N} k_l\lambda_l^{N-1} = \frac{1}{b_N}, \\ \displaystyle\sum_{l=1}^{l_0} k_l\lambda_l^{N} - \sum_{l=l_0+1}^{2N} k_l\lambda_l^{N} = -\frac{b_{N-1}}{b_N^2}, \\ \cdots\cdots\cdots \\ \displaystyle\sum_{l=1}^{l_0} k_l\lambda_l^{2N-1} - \sum_{l=l_0+1}^{2N} k_l\lambda_l^{2N-1} = R(b_{-1}, b_0, \cdots, b_{N-1}, b_N), \end{cases} \tag{10}$$

where $R(b_{-1}, b_0, \cdots, b_{N-1}, b_N)$ is a rational function of $b_{-1}, b_0, \cdots, b_{N-1}, b_N$. If $k_1, \cdots, k_{l_0}$, $-k_{l_0+1}, \cdots, -k_{2N}$ are seen as unknown variables, then the coefficient determinant of the linear system of equations (10) is the Vandermonde determinant $det(\lambda_i^j)$, which is different from zero. A set of values $(k_1^*, \cdots, k_{2N}^*)$ can be uniquely determined from Eq.(10). Therefore, the sequence $\{c_n\}$ defined by

$$c_n = \sum_{l=1}^{l_0} k_l^* \sum_{m\le n-1} \lambda_l^{n-(m+1)} h_m + \sum_{l=l_0+1}^{2N} k_l^* \sum_{m\ge n} \lambda_l^{n-(m+1)} h_m, \qquad n \in Z, \tag{11}$$

is a solution of the difference equation (6).

Secondly, the sequence defined by (11) is a $\widetilde{\mathcal{PAP}}_0$ ($\mathcal{PAP}_0$) sequence. Consider

$$\begin{aligned}\frac{1}{2j}\sum_{n=-j}^{j} |c_n| &= \frac{1}{2j}\sum_{n=-j}^{j} \left| \sum_{l\in L} k_l^* \sum_{m\le n-1} \lambda_l^{n-(m+1)} h_m + \sum_{l\in L'} k_l^* \sum_{m\ge n} \lambda_l^{n-(m+1)} h_m \right| \\ &\le \frac{1}{2j}\sum_{n=-j}^{j}\sum_{l\in L} |k_l^*| \sum_{m\le n-1} |\lambda_l|^{n-(m+1)}|h_m| + \frac{1}{2j}\sum_{n=-j}^{j}\sum_{l\in L'} |k_l^*| \sum_{m\ge n} |\lambda_l|^{n-(m+1)}|h_m| \\ &= \sum_{l\in L}|k_l^*| \left(\frac{1}{2j}\sum_{n=-j}^{j}\sum_{m\le n-1} |\lambda_l|^{n-(m+1)}|h_m| \right) + \sum_{l\in L'}|k_l^*| \left(\frac{1}{2j}\sum_{n=-j}^{j}\sum_{m\ge n} |\lambda_l|^{n-(m+1)}|h_m| \right) \\ &= \sum_{l\in L}|k_l^*|\sigma_1(j) + \sum_{l\in L'}|k_l^*|\sigma_2(j).\end{aligned}$$

We write

$$\begin{aligned}
\sigma_1(j) &= \frac{1}{2j}\sum_{n=-j}^{j}\sum_{m\le n-1}|\lambda_l|^{n-(m+1)}|h_m| \\
&= \frac{1}{2j}\left(\sum_{n=0}^{j}\sum_{m\le n-1}|\lambda_l|^{n-(m+1)}|h_m| + \sum_{n=-j}^{-1}\sum_{m\le n-1}|\lambda_l|^{n-(m+1)}|h_m|\right) \\
&= \frac{1}{2j}\left(\sum_{n=0}^{j}\sum_{m=-1}^{n-1}|\lambda_l|^{n-(m+1)}|h_m| + \sum_{n=0}^{j}\sum_{m=-2}^{-\infty}|\lambda_l|^{n-(m+1)}|h_m|\right. \\
&\quad \left. + \sum_{n=-j}^{-1}\sum_{m=-(j+1)}^{-\infty}|\lambda_l|^{n-(m+1)}|h_m| + \sum_{n=-j+1}^{-1}\sum_{m=n-1}^{-j}|\lambda_l|^{n-(m+1)}|h_m|\right) \\
&= S_1(j)+S_2(j)+S_3(j)+S_4(j),
\end{aligned}$$

and note that $|\lambda_l|<1,\ l\in L$ in $\sigma_1(j)$. Now we prove that $\lim_{j\to+\infty} S_i(j)=0,\ i=1,2,3,4$.

In fact, from Lemma 1 and $f\in\widetilde{\mathcal{P}}\mathcal{AP}_0(R,R)$ ($\mathcal{PAP}_0(R,R)$ or $\widetilde{\mathcal{P}}\mathcal{AP}_0(R,R)\cap M_b(R,R)$), we have

$$\begin{aligned}
S_1(j) &= \frac{1}{2j}\sum_{n=0}^{j}\sum_{m=-1}^{n-1}|\lambda_l|^{n-(m+1)}|h_m| \\
&= \frac{1}{2j}\sum_{m=-1}^{j-1}|h_m|(1+|\lambda_l|+|\lambda_l|^2+\cdots+|\lambda_l|^{j-(m+1)}) \\
&\le \frac{(1-|\lambda_l|)^{-1}}{2j}\sum_{m=-1}^{j-1}|h_m|\to 0 \quad \text{as } j\to+\infty. \\
S_2(j) &= \frac{1}{2j}\sum_{n=0}^{j}\sum_{m=-2}^{-\infty}|\lambda_l|^{n-(m+1)}|h_m| \\
&= \frac{1}{2j}\sum_{m=-2}^{-\infty}|h_m|(|\lambda_l|^{-(m+1)}+|\lambda_l|^{1-(m+1)}+\cdots+|\lambda_l|^{j-(m+1)}) \\
&\le \frac{1}{2j}\sum_{m=-2}^{-\infty}|h_m||\lambda_l|^{-(m+1)}(1-|\lambda_l|)^{-1}.
\end{aligned}$$

Since $\varepsilon(k)=(1/k)\sum_{m=-1}^{-k}|h_m|$ is bounded on Z^+ by using Lemma 1, that is, there exists a $M>0$ such that $\varepsilon(k)<M$, it follows

$$\begin{aligned}
\sum_{m=-2}^{-k}|h_m||\lambda_l|^{-(m+1)} &\le |\lambda_l|^{k-1}k\varepsilon(k)+(|\lambda_l|^{k-2}-|\lambda_l|^{k-1})(k-1)\varepsilon(k-1)+\cdots+(1-|\lambda_l|)\,\varepsilon(1) \\
&\le M(1-|\lambda_l|)^{-1},
\end{aligned}$$

that is, $\sum_{m=-2}^{-\infty}|h_m||\lambda_l|^{-(m+1)}<+\infty$. Therefore, $S_2(j)\to 0$ as $j\to+\infty$.

$$\begin{aligned}
S_3(j) &= \frac{1}{2j}\sum_{n=-j}^{-1}\sum_{m=-(j+1)}^{-\infty}|\lambda_l|^{n-(m+1)}|h_m| \\
&= \frac{1}{2j}\sum_{m=-(j+1)}^{-\infty}|h_m|(|\lambda_l|^{-(m+2)}+\cdots+|\lambda_l|^{-(m+j+1)}) \\
&\le \frac{1}{2j}\sum_{m=-(j+1)}^{-\infty}|h_m||\lambda_l|^{-(m+j+1)}(1-|\lambda_l|)^{-1}.
\end{aligned}$$

Let

$$R_3(k,j) = \sum_{m=-(j+1)}^{-(j+k)} |h_m||\lambda_l|^{-(m+j+1)},$$

$$\varepsilon(k,j) = \frac{1}{k}\sum_{m=-(j+1)}^{-(j+k)} |h_m|.$$

Then for each fixed j, $\lim_{k\to+\infty}\varepsilon(k,j)=0$. Hence the sequence

$$\{M(j):\ M(j) = \sup_{k\in Z^+}\varepsilon(k,j)\}$$

is well defined. Since

$$\varepsilon(k,j) = \frac{k+j}{k}\,\varepsilon(k+j,0) - \frac{j}{k}\,\varepsilon(j,0),$$

and

$$\lim_{j\to+\infty}\varepsilon(k+j,0) = \lim_{j\to+\infty}\frac{1}{k+j}\sum_{m=-1}^{-(k+j)}|h_m| = 0 \quad \text{uniformly for } k\in Z^+,$$

then

$$\lim_{j\to+\infty}\sup_{k\in Z^+}\varepsilon(k+j,0) = \lim_{j\to+\infty}\sup_{u\geq j+1}\varepsilon(u,0) = 0,$$

and $(k+j)/(kj)\leq 2$ for $k\in Z^+$, $j\in Z^+$, we have

$$0\leq \lim_{j\to+\infty}\frac{M(j)}{2j} \leq \lim_{j\to+\infty}\sup_{k\in Z^+}(\varepsilon(k+j,0)+\varepsilon(j,0)) = 0.$$

Therefore

$$\begin{aligned} R_3(k,j) &= |\lambda_l|^{k-1}k\,\varepsilon(k,j) + (|\lambda_l|^{k-2}-|\lambda_l|^{k-1})(k-1)\,\varepsilon(k-1,j) \\ &\quad + \cdots + (1-|\lambda_l|)\,\varepsilon(1,j) \\ &\leq M(j)(1+|\lambda_l|+|\lambda_l|^2+\cdots+|\lambda_l|^{k-1}) \leq M(j)(1-|\lambda_l|)^{-1}, \end{aligned}$$

that is,

$$R_3(+\infty,j) = \sum_{-(j+1)}^{-\infty}|h_m||\lambda_l|^{-(m+j+1)} \leq M(j)(1-|\lambda_l|)^{-1}.$$

So we get that

$$\begin{aligned} \lim_{j\to+\infty} S_3(j) &\leq (1-|\lambda_l|)^{-1}\lim_{j\to+\infty}\frac{1}{2j}R_3(+\infty,j) \\ &\leq (1-|\lambda_l|)^{-2}\lim_{j\to+\infty}\frac{1}{2j}M(j) = 0. \\ S_4(j) &= \frac{1}{2j}\sum_{n=-j+1}^{-1}\sum_{m=n-1}^{-j}|\lambda_l|^{n-(m+1)}|h_m| \\ &= \frac{1}{2j}\sum_{m=-j}^{-2}|h_m|(1+|\lambda_l|+\cdots+|\lambda_l|^{-(m+2)}) \\ &\leq \frac{1}{2j}\sum_{m=-j}^{-2}|h_m|(1-|\lambda_l|)^{-1}\to 0 \quad \text{as } j\to+\infty. \end{aligned}$$

Hence, it follows that $\lim_{j\to+\infty}\sigma_1(j)=0$.

Similarly, it can be deduced that

$$\lim_{j\to+\infty}\sigma_2(j)=\lim_{j\to+\infty}\frac{1}{2j}\sum_{n=-j}^{j}\sum_{m\geq n}|\lambda_l|^{n-(m+1)}|h_m|=0.$$

From Definition 4, we know that the sequence $\{c_n\}$ is a $\widetilde{\mathcal{P}}\mathcal{AP}_0$ ($\mathcal{PAP}_0$) sequence.

Thirdly, Eq.(1) has a $\widetilde{\mathcal{P}}\mathcal{AP}_0(R,R)$ ($\mathcal{PAP}_0(R,R)$) solution. In fact, since the sequence $\{c_n\}_{n\in Z}$ defined by (11) is a $\widetilde{\mathcal{P}}\mathcal{AP}_0$ ($\mathcal{PAP}_0$) sequence and $m(|f|)=0$, for $x(t)$ defined by (4) and $j\geq 1$, $T\in[j-1,j)$, we have

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T}|x(t)|dt &\leq \frac{e^{|a|}}{2T}\sum_{n=-j}^{j}|c_n|+(e^{|a|}+1)\sum_{i=-N}^{N}|a^{-1}a_i|\left(\frac{1}{2T}\sum_{n=-j}^{j}|c_{n+i}|\right)+\frac{e^{|a|}}{2T}\int_{-j-1}^{j}|f(t)|dt\\ &\leq \frac{e^{|a|}}{2(j-1)}\sum_{n=-j}^{j}|c_n|+(e^{|a|}+1)\sum_{i=-N}^{N}|a^{-1}a_i|\left(\frac{j+N}{j-1}\right)\left(\frac{1}{2(j+N)}\sum_{n=-(j+N)}^{j+N}|c_n|\right)\\ &\quad+\left(\frac{e^{|a|}(j+1)}{j-1}\right)\left(\frac{1}{2(j+1)}\int_{-(j+1)}^{j+1}|f(t)|dt\right)\to 0 \quad \text{as } j\to+\infty.\end{aligned}$$

It follows that $x\in\widetilde{\mathcal{P}}\mathcal{AP}_0(R,R)$ ($\mathcal{PAP}_0(R,R)$).

Finally, if $\bar{x}\in\mathcal{PAP}_0(R,R)$ is another solution of Eq.(1) in the case $f\in\mathcal{PAP}_0(R,R)$ or $f\in\widetilde{\mathcal{P}}\mathcal{AP}_0(R,R)\cap M_b$, then $\bar{x}-x\in\mathcal{PAP}_0(R,R)$ is a solution of the corresponding homogeneous equation. Thus, $\{\bar{x}(n)-x(n)\}_{n\in Z}$ is a solution of the homogeneous difference equation (7). Hence, there exist $k_1,\cdots,k_{2N}$ such that

$$\bar{x}(n)-x(n)=k_1\lambda_1^n+\cdots+k_{2N}\lambda_{2N}^n,\qquad n\in Z.$$

From the boundedness of functions in $\mathcal{PAP}_0(R,R)$ or $\widetilde{\mathcal{P}}\mathcal{AP}_0(R,R)\cap M_b$, it follows that $\bar{x}(n)-x(n)\equiv 0$, $n\in Z$. Clearly, this implies $\bar{x}(t)\equiv x(t)$, $t\in R$. This completes the proof of part (1).

(2) Let $f\in\widetilde{\mathcal{P}}\mathcal{AP}(R,R)$ ($\mathcal{PAP}(R,R)$ or $\widetilde{\mathcal{P}}\mathcal{AP}(R,R)\cap M_b(R,R)$). Then $f=f_1+f_0$, where f_1 is an almost periodic function on R, $f_0\in\widetilde{\mathcal{P}}\mathcal{AP}_0(R,R)$ ($\mathcal{PAP}_0(R,R)$ or $\widetilde{\mathcal{P}}\mathcal{AP}_0(R,R)\cap M_b(R,R)$). According to Theorem 1 in [**14**], we know that the differential equation

$$x'=a\,x(t)+\sum_{i=-N}^{N}a_i\,x([t+i])+f_1(t)$$

has a unique almost periodic solution $x_1(t)$. From the previous part, it follows that the differential equation

$$x'=a\,x(t)+\sum_{i=-N}^{N}a_i\,x([t+i])+f_0(t)$$

has a solution $x_0\in\widetilde{\mathcal{P}}\mathcal{AP}_0(R,R)$ ($\mathcal{PAP}_0(R,R)$). And x_0 is unique in the case $f_0\in\mathcal{PAP}_0(R,R)$ or $f_0\in\widetilde{\mathcal{P}}\mathcal{AP}_0(R,R)\cap M_b$. Therefore, the differential equation (1) has a solution $x=x_1+x_0\in\widetilde{\mathcal{P}}\mathcal{AP}(R,R)$ ($\mathcal{PAP}(R,R)$). And $x=x_1+x_0$ is unique in the case $f\in\mathcal{PAP}(R,R)$ or $f\in\widetilde{\mathcal{P}}\mathcal{AP}(R,R)\cap M_b$. This completes the proof of the Theorem 1.

Before we give the proof of Theorem 2, we want to show the following results.

Lemma 2. *$E_0 = \{\varphi : \varphi \in \mathcal{PAP}_0(R,R)$ and $\lim_{n\to+\infty}(1/2n)\sum_{k=-n}^{n}|\varphi(k)| = 0\}$ is a closed subset of $\mathcal{PAP}_0(R,R)$.*

Proof. For any $\varphi \in \overline{E}_0$, which is the closure of E_0, it follows from [**15**] that $\varphi \in \mathcal{PAP}_0(R,R)$ and there exists a sequence $\{\varphi_m\}_{m\in Z^+} \subset E_0$ such that $\lim_{m\to+\infty}||\varphi_m - \varphi|| = 0$, in particular, $\{|\varphi_m(t)|\}$ converges uniformly to $|\varphi(t)|$ on R. Thus

$$\begin{aligned}\lim_{n\to+\infty}\frac{1}{2n}\sum_{k=-n}^{n}|\varphi(k)| &= \lim_{n\to+\infty}\frac{1}{2n}\sum_{k=-n}^{n}(\lim_{m\to+\infty}|\varphi_m(k)|)\\ &= \lim_{m\to+\infty}\left(\lim_{n\to+\infty}\frac{1}{2n}\sum_{k=-n}^{n}|\varphi_m(k)|\right) = 0,\end{aligned}$$

that is, $\varphi \in E_0$. This completes the proof.

From this lemma and the properties of almost periodic functions in ref. [**10**], we have

Lemma 3. *$E = \{\varphi : \varphi \in \mathcal{PAP}(R,R)$, and its ergodic perturbation $\varphi_0 \in E_0\}$ is a closed subset of $\mathcal{PAP}(R,R)$.*

Lemma 4. *Suppose that $x \in E_0$ and $g(t,\cdot,\cdot) \in \widetilde{\mathcal{P}}\mathcal{AP}_0(R\times R^2,R)$ and satisfies (3). Then the function $g(t,x(t),x([t]))$ is a $\widetilde{\mathcal{P}}\mathcal{AP}_0$ function on R.*

Proof. From the Lipschitz condition (3), for $T \in [n,n+1)$, $n \in Z^+$, we have

$$\begin{aligned}0 &\le \frac{1}{2T}\int_{-T}^{T}|g(t,x(t),x([t]))dt\\ &\le \frac{1}{2T}\int_{-T}^{T}|g(t,0,0)|dt + \frac{\eta}{2T}\int_{-T}^{T}|x(t)|dt + \left(\frac{n+1}{n}\right)\frac{\eta}{2(n+1)}\sum_{k=-(n+1)}^{n+1}|x(k)| \to 0 \quad \text{as}\, T\to+\infty.\end{aligned}$$

It follows that $g(t,x(t),x([t]))$ is a $\widetilde{\mathcal{P}}\mathcal{AP}_0$ function on R.

Lemma 5. *Let $g \in \widetilde{\mathcal{P}}\mathcal{AP}(R\times\Omega,R)$, $\varphi \in \widetilde{\mathcal{P}}\mathcal{AP}(R,R)$, and $g = g_0 + g_1$, $\varphi = \varphi_0 + \varphi_1$, where g_0 (g_1) and φ_0 (φ_1) are the ergodic perturbations (almost periodic components) of g and φ respectively, $\Omega \subset R^2$ and $\varphi_1(R)\times\varphi_1(Z) \subset \Omega$. If g satisfies the Lipschitz condition (3), then $g_0(\cdot,\varphi_1(\cdot),\varphi_1([\cdot])) \in \widetilde{\mathcal{P}}\mathcal{AP}_0(R,R)$.*

Proof. Since $\varphi_1(R)$ is bounded and the uniformly almost periodic function g_1 is uniformly continuous, for any $\varepsilon > 0$, one can find m $(m < +\infty)$ open balls O_k with center $(z_1^{(k)}, z_2^{(k)}) \in \Omega$, $k = 1,2,\cdots,m$, and radius less than $\varepsilon/(6\eta)$ such that

$$\varphi_1(R)\times\varphi_1(Z) \subset \cup_{k=1}^{m} O_k$$

and

$$|g_1(t,z_1,z_2) - g_1(t,z_1^{(k)},z_2^{(k)})| < \frac{\varepsilon}{3}, \quad z=(z_1,z_2)\in O_k,\ t\in R. \tag{12}$$

The set

$$B_k = \{t\in R,\ (\varphi_1(t),\varphi_1([t]))\in O_k\} \tag{13}$$

is open and $R = \cup_{k=1}^{m} B_k$. Let $E_k = B_k - \cup_{j=1}^{k-1}B_j$. Then $E_k\cap E_j = \emptyset$ when $k\ne j$, $1\le k,\ j\le m$.

It follows from the fact that each $g_0(\cdot,z_1^{(k)},z_2^{(k)}) \in \widetilde{\mathcal{P}}\mathcal{AP}_0(R,R)$, that there exists $T_0 > 0$ such that

$$\sum_{k=1}^{m}\frac{1}{2T}\int_{-T}^{T}|g_0(t,z_1^{(k)},z_2^{(k)})|dt < \frac{\varepsilon}{3}, \quad T > T_0. \tag{14}$$

From (3),(12),(13) and (14), we have

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T}|g_0(t,\varphi_1(t),\varphi_1([t]))|dt\\
&\le \frac{1}{2T}\sum_{k=1}^{m}\int_{E_k\cap[-T,T]}(|g_0(t,\varphi_1(t),\varphi_1([t]))-g_0(t,z_1^{(k)},z_2^{(k)})|+|g_0(t,z_1^{(k)},z_2^{(k)})|)dt\\
&\le \frac{1}{2T}\sum_{k=1}^{m}\int_{E_k\cap[-T,T]}(|g(t,\varphi_1(t),\varphi_1([t]))-g(t,z_1^{(k)},z_2^{(k)})|\\
&\quad+|g_1(t,\varphi_1(t),\varphi_1([t]))-g_1(t,z_1^{(k)},z_2^{(k)})|+|g_0(t,z_1^{(k)},z_2^{(k)})|)dt\\
&\le \frac{1}{2T}\sum_{k=1}^{m}\int_{E_k\cap[-T,T]}(\eta|\varphi_1(t)-z_1^{(k)}|+\eta|\varphi_1([t])-z_2^{(k)}|\\
&\quad+|g_1(t,\varphi_1(t),\varphi_1([t]))-g_1(t,z_1^{(k)},z_2^{(k)})|+|g_0(t,z_1^{(k)},z_2^{(k)})|)dt<\varepsilon,\quad T>T_0.
\end{aligned}$$

The proof is finished.

Proof of Theorem 2.

(1) For any $\phi\in E_0$, the following equation

$$x'(t)=a\,x(t)+\sum_{i=-N}^{N}a_i\,x([t+i])+g(t,\phi(t),\phi([t])), \tag{15}$$

has a unique solution $\mathcal{T}\phi\in\mathcal{PAP}_0(R,R)$ by using Theorem 1 (g is bounded) and Lemma 4. Since $\{c_n\}_{n\in Z}$, defined by (11), is a $\mathcal{PAP}_0$ sequence in the proof of Theorem 1, it follows that $\mathcal{T}$ is a mapping from E_0 into itself, where E_0 is the same as in Lemma 2. For any $\phi,\psi\in E_0$, $\mathcal{T}\phi-\mathcal{T}\psi$ satisfies the following equation

$$z'(t)=a\,z(t)+\sum_{i=-N}^{N}a_i\,z([t+i])+g(t,\phi(t),\phi([t]))-g(t,\psi(t),\psi([t])).$$

Thus we get

$$\sum_{i=-N}^{N}b_i[(\mathcal{T}\phi)(n+i)-(\mathcal{T}\psi)(n+i)]=H_n,$$

where

$$H_n=-\int_n^{n+1}e^{a(n+1-s)}[g(s,\phi(s),\phi([s]))-g(s,\psi(s),\psi([s]))]ds.$$

It follows from the proof of Theorem 1 that there exist constants $k_1^*,\cdots,k_{2N}^*$ such that

$$(\mathcal{T}\phi)(n)-(\mathcal{T}\psi)(n)=\sum_{l=1}^{l_0}k_l^*\sum_{m\le n-1}\lambda_l^{n-(m+1)}H_m+\sum_{l=l_0+1}^{2N}k_l^*\sum_{m\ge n}\lambda_l^{n-(m+1)}H_m.$$

Hence, there exists $K_0>0$, such that

$$\begin{aligned}
|(\mathcal{T}\phi)(n)-(\mathcal{T}\psi)(n)|&\le K_0\sup_{m\in Z}|H_m|\\
&\le 2e^{|a|}K_0\,\eta\,|\phi-\psi|,\qquad \forall n\in Z.
\end{aligned}$$

It follows that

$$|\mathcal{T}\phi(t)-\mathcal{T}\psi(t)|\le\left\{2e^{|a|}K_0\left(e^{|a|}+(e^{|a|}+1)\sum_{i=-N}^{N}|a^{-1}a_i|\right)+2e^{|a|}\right\}\eta\,|\phi-\psi|,$$

$\forall t \in R$. Hence, there exists $\eta_* > 0$, such that if $0 \le \eta < \eta_*$, $\mathcal{T} : E_0 \to E_0$ is a contracting mapping. This implies that there exists a unique $\phi \in E_0$ such that $\mathcal{T}\phi = \phi$. The proof of part (1) is finished.

(2) Let g_1 and g_0 be the almost periodic component and the ergodic perturbation, respectively, of the function g. For every $\varphi \in E$, where E is the same as in Lemma 3, let $\varphi = \varphi_1 + \varphi_0$, where φ_1 and φ_0 are the almost periodic component and the ergodic perturbation, respectively, of the function φ. Then the equation

$$x'(t) = a\,x(t) + \sum_{i=-N}^{N} a_i\,x([t+i]) + g_1(t, \varphi_1(t), \varphi_1([t])) \tag{16}$$

has a unique almost periodic solution x_1 by using Lemma 5 in [**14**] and Theorem 1 in [**14**].

On the other hand, from Lemma 5, we have

$$\begin{aligned}
&\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} |g(t,\varphi(t),\varphi([t])) - g_1(t,\varphi_1(t),\varphi_1([t]))|dt \\
&\le \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} (|g(t,\varphi(t),\varphi([t])) - g(t,\varphi_1(t),\varphi_1([t]))| + |g_0(t,\varphi_1(t),\varphi_1([t]))|)dt \\
&\le \lim_{T\to+\infty} \frac{\eta}{2T}\int_{-T}^{T} |\varphi_0(t)|dt + \lim_{T\to+\infty} \frac{[T]+1}{[T]}\frac{1}{2([T]+1)} \sum_{k=-[T]-1}^{[T]+1} |\varphi_0(k)| \\
&\quad + \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} |g_0(t,\varphi_1(t),\varphi_1([t]))|dt = 0,
\end{aligned} \tag{17}$$

i.e., $g(\cdot,\varphi(\cdot),\varphi([\cdot])) - g_1(\cdot,\varphi_1(\cdot),\varphi_1([\cdot])) \in \widetilde{\mathcal{PAP}}_0(R,R)$. According to Theorem 1, we know that the equation

$$x'(t) = a\,x(t) + \sum_{i=-N}^{N} a_i\,x([t+i]) + g(t,\varphi(t),\varphi([t])) - g_1(t,\varphi_1(t),\varphi_1([t])) \tag{18}$$

has a unique solution $x_0 \in E_0$. Hence the equation

$$x'(t) = a\,x(t) + \sum_{i=-N}^{N} a_i\,x([t+i]) + g(t,\varphi(t),\varphi([t]))$$

has a unique solution $\mathcal{T}\varphi = x_0 + x_1 \in E$, that is, a mapping $\mathcal{T}$ from E to itself is obtained. Similar to the estimation in the proof of part 1, there exists $\eta_* > 0$ such that when $0 < \eta < \eta_*$, $\mathcal{T} :\ E \to E$ is a contracting mapping. This implies that $\mathcal{T}$ has a unique fixed point $\varphi \in E$ such that $\mathcal{T}\varphi = \varphi$. The proof of this theorem is finished.

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