NON-ATKINSON PERTURBATIONS OF NONAUTONOMOUS LINEAR HAMILTONIAN SYSTEMS: EXPONENTIAL DICHOTOMY AND NONOSCILLATION

CARMEN NÚÑEZ AND RAFAEL OBAYA

Abstract. We analyze the presence of exponential dichotomy (ED) and of global existence of Weyl functions $M^{\pm}$ for one-parametric families of finite-dimensional nonautonomous linear Hamiltonian systems defined along the orbits of a compact metric space, which are perturbed from an initial one in a direction which does not satisfy the classical Atkinson condition: either they do not have ED for any value of the parameter; or they have it for at least all the nonreal values, in which case the Weyl functions exist and are Herglotz. When the parameter varies in the real line, and if the unperturbed family satisfies the properties of exponential dichotomy and global existence of M^+, then these two properties persist in a neighborhood of 0 which agrees either with the whole real line or with an open negative half-line; and in this last case, the ED fails at the right end value. The properties of ED and of global existence of M^+ are fundamental to guarantee the solvability of classical minimization problems given by linear-quadratic control processes.

1. Introduction

The theory of exponential dichotomy has played a central role in the study of finite and infinite dimensional dynamical systems, including those arising in the analysis of nonautonomous differential equations. In the linear case, the occurrence of exponential dichotomy is directly connected with the invertibility of the associated operators. And, in the nonlinear case, the robustness of the exponential dichotomy of the linearized flows converts this property in an essential tool to analyze the behavior of the solutions.

In particular, the exponential dichotomy is also fundamental in the description of invariant manifolds, perturbation problems, bifurcation patterns, homoclinic trajectories and spectral theory, among many other questions. The works of Coppel [6, 7], Massera and Schaeffer [31], Hale [18], Sacker and Sell [39, 40, 41, 42, 43, 44], Sell [45], Chow and Hale [3], Palmer [33, 34], Johnson [20], Vanderbauwhede and van Gils [51], Vanderbauwhede [50], Henry [19], Johnson and Yi [28], Chow and Leiva [4, 5], Shen and Yi [49], Chicone and Latushkin [2], Pliss and Sell [35], and Johnson *et al.* [27] (which compose an incomplete list) provide an exhaustive analysis of all these topics.

In the classical field of finite-dimensional linear Hamiltonian differential equations with periodic time-dependent coefficients, the existence and robustness of the exponential dichotomy is directly related to the regions of instability and total instability studied by Gel'fand and Lidskiĭ [16] and Yakubovich [52, 53]. In the more

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general setting of nonautonomous linear Hamiltonian systems with bounded and uniformly continuous coefficients, these questions are extensively analyzed in the book Johnson *et al.* [27], which presents a unified version of many previous works due to the authors of the book and to many other researchers. In particular, in this book, the applicability of the exponential dichotomy results to the study of nonautonomous linear-quadratic control processes is extensively analyzed. We will now explain briefly a point of this analysis, which is central in order to understand the scope of the present paper.

Let us consider the control problem

$$\mathbf{x}' = A_0(t)\,\mathbf{x} + B_0(t)\,\mathbf{u}\,, \tag{1.1}$$

where $\mathbf{x}\in\mathbb{R}^n$ and $\mathbf{u}\in\mathbb{R}^m$, together with the quadratic form (supply rate)

$$\mathcal{Q}(t,\mathbf{x},\mathbf{u}) := \frac{1}{2}\left(\langle \mathbf{x}, G_0(t)\,\mathbf{x}\rangle + 2\langle \mathbf{x}, g_0(t)\,\mathbf{u}\rangle + \langle \mathbf{u}, R_0(t)\,\mathbf{u}\rangle\right).$$

The functions A_0, B_0, G_0, g_0, and R_0 are assumed to be bounded and uniformly continuous functions on $\mathbb{R}$, with values in the sets of real matrices of the appropriate dimensions. In addition, G and R are symmetric, and $R(t)\ge \rho I_m$ for a common $\rho>0$ and all $t\in\mathbb{R}$. We also fix $\mathbf{x}_0\in\mathbb{R}^n$ and introduce the quadratic functional

$$\mathcal{J}_{\mathbf{x}_0}(\mathbf{x},\mathbf{u}) := \int_0^\infty \mathcal{Q}(t,\mathbf{x}(t),\mathbf{u}(t))\,dt$$

evaluated on the so-called *admissible pairs* $(\mathbf{x},\mathbf{u})\colon[0,\infty)\to\mathbb{R}^n\times\mathbb{R}^m$; i.e. those for which $\mathbf{u}$ belongs to $L^2((0,\infty),\mathbb{R}^m)$ and the solution $\mathbf{x}(t)$ of (1.1) for this control with $\mathbf{x}(0)=\mathbf{x}_0$ belongs to $L^2((0,\infty),\mathbb{R}^n)$. The problem to consider is that of minimizing $\mathcal{J}_{\mathbf{x}_0}$ relative to the set of admissible pairs.

By means of a standard construction (the so called hull or Bebutov construction), this problem can be included in a family, given by the control problems

$$\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\mathbf{u} \tag{1.2}$$

and by the functionals

$$\begin{aligned}\mathcal{Q}_\omega(t,\mathbf{x},\mathbf{u}) &:= \frac{1}{2}\left(\langle \mathbf{x}, G(\omega{\cdot}t)\,\mathbf{x}\rangle + 2\langle \mathbf{x}, g(\omega{\cdot}t)\,\mathbf{u}\rangle + \langle \mathbf{u}, R(\omega{\cdot}t)\,\mathbf{u}\rangle\right),\\ \mathcal{J}_{\mathbf{x}_0,\omega}(\mathbf{x},\mathbf{u}) &:= \int_0^\infty \mathcal{Q}_\omega(t,\mathbf{x}(t),\mathbf{u}(t))\,dt\end{aligned}$$

for $\omega\in\Omega$ and $\mathbf{x}_0\in\mathbb{R}^n$. Here, Ω is a compact metric space admitting a continuous flow, $\omega{\cdot}t$ is the orbit of a point $\omega\in\Omega$, A, B, G, g, and R are bounded and uniformly continuous matrix-valued functions on Ω, G and R are symmetric, and $R>0$. It is important to point out that Ω is minimal in the case of recurrence of the initial coefficients, which includes the autonomous, periodic, quasi-periodic, almost-periodic and almost-automorphic cases.

The Pontryagin Maximum Principle relates the problem of minimizing $\mathcal{J}_{\mathbf{x}_0,\omega}$ to the properties of the family of linear Hamiltonian systems

$$\mathbf{z}' = H(\omega{\cdot}t)\,\mathbf{z}\,,\qquad \omega\in\Omega\,, \tag{1.3}$$

where $\mathbf{z}=\left[\begin{smallmatrix}\mathbf{x}\\ \mathbf{y}\end{smallmatrix}\right]$ for $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$ and

$$H = \begin{bmatrix} A - B\,R^{-1}g^T & B\,R^{-1}B^T\\ G - g\,R^{-1}g^T & -A^T + g\,R^{-1}B^T\end{bmatrix}.$$

More precisely, under a certain uniform stabilization condition, it turns out that the minimization problem for each one of the functionals $\mathcal{J}_{\omega,\mathbf{x}_0}$ is solvable if the family (1.3) admits exponential dichotomy and if, in addition, for every $\omega \in \Omega$, the Lagrange plane $l^+(\omega)$ composed by those initial data $\mathbf{z}_0$ giving rise to a solutions bounded at $+\infty$ admits a basis whose vectors compose the columns of a matrix $\left[\begin{smallmatrix} I_n \\ M^+(\omega) \end{smallmatrix}\right]$. In other words, if the family admits exponential dichotomy (or the *frequency condition* holds) and the Weyl function M^+ globally exists (or the *nonoscillation condition* is satisfied). (For further purposes we point out that the Weyl function M^- is associated in the analogous way to the Lagrange plane $l^-(\omega)$ composed by the initial data $\mathbf{z}_0$ of the solutions bounded at $-\infty$.) In addition, if this is the case, the unique minimizing pair $(\widetilde{\mathbf{x}}(t), \widetilde{\mathbf{u}}(t))$ for $\mathcal{J}_{\omega,\mathbf{x}_0}$ is determined from the solution $\left[\begin{smallmatrix} \widetilde{\mathbf{x}}(t) \\ \widetilde{\mathbf{y}}(t) \end{smallmatrix}\right]$ of (1.3) with initial data $\left[\begin{smallmatrix} \mathbf{x}_0 \\ M^+(\omega)\,\mathbf{x}_0 \end{smallmatrix}\right]$ by means of the feedback rule $\widetilde{\mathbf{u}}(t) = R^{-1}(\omega{\cdot}t)\,B^T(\omega{\cdot}t)\,\widetilde{\mathbf{y}}(t) - R^{-1}(\omega{\cdot}t)\,g^T(\omega{\cdot}t)\,\widetilde{\mathbf{x}}(t)$. And, as a matter of fact, both situations (solvability and "frequency plus nonoscillation conditions") are equivalent in many dynamical situations, as in the case of minimality of Ω. This result, first published in Fabbri *et al.* [13] and [10] (and which is extremely detailed in Chapter 7 of [27]), constitutes a nonautonomous version of the Yakubovich Frequency Theorem for the periodic case, which appears in [52, 53].

The historical and practical importance of the above result justifies the interest of this paper, whose central goal is to analyze the presence and preservation of the exponential dichotomy and the nonoscillation condition in parametric families of linear Hamiltonian systems.

In what follows, we explain simultaneously the structure of the paper and its main achievements. Section 2 summarizes some basic notions on topological dynamics, and explains with some detail the concepts of exponential dichotomy, uniform weak disconjugacy, and rotation number, which are fundamental in the statements and proofs of the main results.

From now on, Ω is a compact metric space with a continuous flow, and we represent by $\{\omega{\cdot}t \mid t \in \mathbb{R}\}$ the orbit of the element $\omega \in \Omega$. In addition, H_1, H_2, H_3 and Δ are continuous $n \times n$ matrix-valued functions on Ω, and H_2, H_3 and Δ take symmetric values.

In Section 3 we consider the families of linear Hamiltonian systems

$$\mathbf{z}' = H^\lambda(\omega{\cdot}t)\,\mathbf{z}\,, \quad \text{where } H^\lambda := \begin{bmatrix} H_1 & H_3 - \lambda\Delta \\ H_2 & -H_1^T \end{bmatrix} \tag{1.4}$$

for $\omega \in \Omega$. The parameter λ varies in $\mathbb{C}$. If the matrix-valued function $\Gamma := \left[\begin{smallmatrix} 0_n & 0_n \\ 0_n & \Delta \end{smallmatrix}\right]$ satisfies the so-called Atkinson definiteness condition (see Atkinson [1]), then the systems $(1.4)^\lambda$ satisfy the frequency and admit both Weyl functions at least for $\lambda \in \mathbb{C} - \mathbb{R}$. This is an already classical result due to Johnson [21]. Here, we analyze the problem for $\Delta > 0$ without imposing the Atkinson hypothesis, and prove that two dynamical possibilities arise: either the families $(1.4)^\lambda$ have exponential dichotomy for (at least) all $\lambda \in \mathbb{C} - \mathbb{R}$, in which case the Weyl functions $M^\pm(\omega, \lambda)$ globally exist and are Herglotz functions; or the family $(1.4)^\lambda$ does not have exponential dichotomy for any $\lambda \in \mathbb{C}$, which turns out to be equivalent to the existence of a point $\omega \in \Omega$ and a nonzero bounded function of the form $\mathbf{z}(t, \omega) = \left[\begin{smallmatrix} \mathbf{z}_1(t,\omega) \\ \mathbf{0} \end{smallmatrix}\right]$ which solves the system $(1.4)^\lambda$ corresponding to ω for all $\lambda \in \mathbb{C}$. These results (excepting the existence of Weyl functions) are easily transferable to the families

$$\mathbf{z}' = \widetilde{H}^\lambda(\omega{\cdot}t)\,\mathbf{z}\,,\quad \text{where}\ \ \widetilde{H}^\lambda := \begin{bmatrix} H_1 & H_3 \\ H_2-\lambda\Delta & -H_1^T \end{bmatrix}. \tag{1.5}$$

In this case, the second dynamical situation is equivalent to the existence of a point $\omega\in\Omega$ and a nonzero bounded function of the form $\mathbf{z}(t,\omega)=\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t,\omega)\end{smallmatrix}\right]$ which solves the system $(1.5)^\lambda$ corresponding to ω for all $\lambda\in\mathbb{C}$. In particular, all the systems corresponding to ω are *abnormal* systems. This type of systems have been extensively studied during the last decades: see e.g. Reid [36, 37], Kratz [30], Šepitka and Šimon Hilscher [46, 47, 48] Fabbri *et al.* [15], Johnson *et al.* [24], and references therein.

In Section 4 we go further in the analysis of the families (1.5) with $H_3\ge 0$ and $\Delta>0$. More precisely, we assume that $(1.5)^0$ admits exponential dichotomy (ED) and satisfies the nonoscillation condition (NC), and define

$$\mathcal{I} := \{\lambda\in\mathbb{R}\,|\,(1.5)^\lambda \text{ has ED and satisfies NC}\}\,.$$

Under the assumption of existence of an ergodic measure on Ω with full topological support (which holds at least in the case of the minimality of Ω), we prove among other properties that $\mathcal{I}$ is either the whole line or an open negative half-line; and that, in addition, if $\mathcal{I}=(-\infty,\lambda^*)$ for a real λ^*, then the family $(1.5)^{\lambda^*}$ does not have exponential dichotomy. This result improves and extends a previous theorem of Johnson *et al.* [26]. In its proof a fundamental role is played by the occurrence of uniform weak disconjugacy and by the properties of the rotation number: both properties are fundamental to determine the presence of exponential dichotomy, in different settings. The reader is referred to Johnson *et al.* [25], Fabbri *et al.* [14, 9], Johnson *et al.* [26, 23] and Chapter 5 of [27] for an in-depth analysis of the uniform weak disconjugacy property, and to Johnson [21], Novo *et al.* [32], Fabbri *et al.* [11, 12] and Chapter 2 of [27] for the definition and main properties of the rotation number. The result concerning the shape and properties of $\mathcal{I}$ is finally extended to the case in which the base flow is distal.

This paper is dedicated to the memory of George Sell. Among his large number of achievements, the development of the theory of exponential dichotomies for nonautonomous dynamical systems giving by skew-product flows on vector bundles with compact base, is more than fundamental in the work of the authors of this paper.

2. Preliminaries

All the contents of this preliminary section can be found in Johnson *et al.* [27], where the reader will also find a quite exhaustive list of references for the origin of the results that we summarize here.

Let us begin by establishing some notation. As usual, $\mathbb{R}$ and $\mathbb{C}$ represent the real line and the complex plane. If $\lambda\in\mathbb{C}$, $\operatorname{Re}\lambda$ and $\operatorname{Im}\lambda$ are respectively its real and imaginary parts.

Now let $\mathbb{K}$ represent $\mathbb{R}$ or $\mathbb{C}$. The set $\mathbb{M}_{d\times m}(\mathbb{K})$ is the set of $d\times m$ matrices with entries in $\mathcal{K}$. As usual, $\mathbb{K}^d := \mathbb{M}_{d\times 1}(\mathbb{K})$, and A^T is the transpose of the matrix A. The subset $\mathbb{S}_d(\mathbb{K})\subset\mathbb{M}_{d\times d}(\mathbb{K})$ is composed by the symmetric matrices. If $M\in\mathbb{S}_d(\mathbb{R})$ is symmetric, the expressions $M>0$, $M<0$, $M\ge 0$, and $M\le 0$ mean that it is positive definite, positive semidefinite, negative definite, and negative semidefinite. If Ω is a topological space and $M\colon\Omega\to\mathbb{S}_d(\mathbb{K})$ is a map, $M>0$

means that $M(\omega) > 0$ for all the elements $\omega \in \Omega$, and $M < 0$, $M \geq 0$, and $M \leq 0$ have the analogous meaning. It is also obvious what $M_1 > M_2$, $M_1 \geq M_2$, $M_1 < M_2$, and $M_1 \leq M_2$ mean. We represent by I_d and 0_d the identity and zero $d \times d$ matrices, and by $\mathbf{0}$ the null vector of $\mathbb{K}^d$ for all d. If $\mathbf{z} \in \mathbb{K}^d$, its Euclidean norm is $\|z\|$, and if $A \in \mathbb{M}_{d\times m}(\mathbb{K})$, then $\|A\|$ is the associated operator norm.

A (real or complex) *Lagrange plane* is an n-dimensional (real or complex) linear space such that $\mathbf{z}^T J\,\mathbf{w} = 0$ for any pair of elements $\mathbf{z}$ and $\mathbf{w}$. A Lagrange plane l *is represented by* $\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ if the column vectors of the matrix form a basis of the n-dimensional linear space l. Hence, it can be also represented by $\left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right]$ if and only if $\det L_1 \neq 0$, in which case the matrix $M = L_2 L_1^{-1}$ is symmetric.

Now we will recall some basic concepts and properties on topological dynamics and measure theory. Let Ω be a complete metric space. A (*real and continuous*) *global flow* on Ω is a continuous map $\sigma\colon \mathbb{R}\times\Omega \to \Omega$, $(t,\omega) \mapsto \sigma(t,\omega)$ such that $\sigma_0 = \mathrm{Id}$ and $\sigma_{s+t} = \sigma_t \circ \sigma_s$ for each $s,t \in \mathbb{R}$, where $\sigma_t(\omega) = \sigma(t,\omega)$. The flow is *local* if the map σ is defined, continuous, and satisfies the previous properties on an open subset of $\mathbb{R}\times\Omega$ containing $\{0\}\times\Omega$.

Let (Ω,σ) be a global flow. The *σ-orbit* of a point $\omega \in \Omega$ is the set $\{\sigma_t(\omega) \mid t \in \mathbb{R}\}$. Restricting the time to $t \geq 0$ or $t \leq 0$ provides the definition of *forward* or *backward* σ-semiorbit. A subset $\Omega_1 \subset \Omega$ is *σ-invariant* if $\sigma_t(\Omega_1) = \Omega_1$ for every $t \in \mathbb{R}$. A σ-invariant subset $\Omega_1 \subset \Omega$ is *minimal* if it is compact and does not contain properly any other compact σ-invariant set; or, equivalently, if each one of the two semiorbits of anyone of its elements is dense in it. The continuous flow (Ω,σ) is *minimal* if Ω itself is minimal.

If the set $\{\sigma_t(\omega) \mid t \geq 0\}$ is relatively compact, the *omega limit set* of ω_0 is given by those points $\omega \in \Omega$ such that $\omega = \lim_{m\to\infty} \sigma(t_m,\omega_0)$ for some sequence $(t_m) \uparrow \infty$. This set is nonempty, compact, connected and σ-invariant. The definition and properties of the *alpha limit set* of ω_0 are analogous, working now with sequences $(t_m) \downarrow -\infty$.

Let m be a normalized Borel measure on Ω; i.e. a finite regular measure defined on the Borel subsets of Ω and with $m(\Omega) = 1$. The measure m is *σ-invariant* if $m(\sigma_t(\Omega_1)) = m(\Omega_1)$ for every Borel subset $\Omega_1 \subset \Omega$ and every $t \in \mathbb{R}$. If, in addition, $m(\Omega_1) = 0$ or $m(\Omega_1) = 1$ for every σ-invariant subset $\Omega_1 \subset \Omega$, then the measure m is *σ-ergodic*. A real continuous flow (Ω,σ) admits at least an ergodic measure. And the *topological support* of m, $\operatorname{Supp} m$, is the complement of the largest open set $O \subset \Omega$ for which $m(O) = 0$. In the case that Ω is minimal, then it agrees with the topological support of any σ-ergodic measure.

In the rest of the paper, (Ω,σ) will be a real continuous global flow on a compact metric space, and we will denote $\omega{\cdot}t = \sigma(t,\omega)$. We represent by $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$. Our starting point is the family of linear Hamiltonian systems

$$\mathbf{z}' = H(\omega{\cdot}t)\,\mathbf{z}\,, \quad \omega \in \Omega \tag{2.1}$$

where $H\colon \Omega \to \mathfrak{sp}(n,\mathbb{K})$ is continuous. Here, $\mathfrak{sp}(n,\mathbb{K})$ is the Lie algebra of the infinitesimally symplectic matrices,

$$\mathfrak{sp}(n,\mathbb{K}) := \{H \in \mathbb{M}_{2n\times 2n}(\mathbb{K}) \mid H^T J + JH = 0_{2n}\}$$

where $J = \left[\begin{smallmatrix} 0_n & -I_n \\ I_n & 0_n \end{smallmatrix}\right]$, so that H takes the form

$$H(\omega) = \begin{bmatrix} H_1(\omega) & H_3(\omega) \\ H_2(\omega) & -H_1^T(\omega) \end{bmatrix},$$

with $H_2^T = H_2$ and $H_3^T = H_3$. Let $U(t,\omega)$ denote the fundamental matrix solution of the system (2.1) for $\omega \in \Omega$ with $U(0,\omega) = I_{2n}$. The family (2.1) induces a real continuous global flow on the linear bundle $\Omega \times \mathbb{K}^{2n}$, given by

$$\tau_{\mathbb{K}} \colon \mathbb{R} \times \Omega \times \mathbb{K}^{2n} \to \Omega \times \mathbb{K}^{2n}\,, \quad (t,\omega,\mathbf{z}) \mapsto (\omega{\cdot}t, U(t,\omega)\,\mathbf{z})\,. \tag{2.2}$$

This flow is called of *skew-product type* since its first component agrees with the base flow, and *linear* since the second component is a linear map for each $\omega \in \Omega$. Frequently, a family of this type comes from a single nonautonomous Hamiltonian system $\mathbf{z}' = H_0(t)\,\mathbf{z}$ by means of the well known Bebutov construction: if H_0 is bounded and uniformly continuous on $\mathbb{R}$, then its *hull* Ω, which is defined by $\Omega := \mathrm{cls}\{H_t \mid t \in \mathbb{R}\}$ (where $H_t(s) = H_0(t+s)$ and the closure is taken in the compact-open topology), is a compact metric space; and the time-translation defines a continuous flow σ on it. The base space Ω can hence be understood as the space in which the nonautonomous law varies with respect to time. Under additional recurrence properties on H_0, the base flow is minimal. This is the case if H_0 is almost periodic or almost automorphic. Weaker conditions on H_0 may provide a non minimal hull, which can contain different minimal subsets. In some of these cases the solutions of the different linear Hamiltonian systems of the family may show a significatively different qualitative behavior.

However, we will not assume that the family (2.1) comes from a single equation by means of the Bebutov construction, which makes our analysis more general.

In the rest of this section we recall some basic concepts and some associated properties related to families of the form (2.1). The analysis contained in this paper either concerns these properties (this is the case of the *exponential dichotomy*, *nonoscillation condition*, and *uniform weak disconjugacy*) or requires them as tools for the proofs (as in the case of the *rotation number*).

Definition 2.1. The family (2.1) has *exponential dichotomy* (or *ED* for short) *over* Ω if there exist constants $\eta \ge 1$ and $\beta > 0$ and a splitting $\Omega \times \mathbb{K}^{2n} = L^+ \oplus L^-$ of the bundle into the Whitney sum of two closed subbundles such that

- L^+ and L^- are invariant under the flow $\tau_{\mathbb{K}}$ given by (2.2) on $\Omega \times \mathbb{K}^{2n}$; that is, if $(\omega,\mathbf{z})$ belongs to L^+ (or to L^-), so does $(\omega{\cdot}t, U(t,\omega)\,\mathbf{z})$ for all $t \in \mathbb{R}$.
- $\|U(t,\omega)\,\mathbf{z}\| \le \eta\, e^{-\beta t}\,\|\mathbf{z}\|$ for every $t \ge 0$ and $(\omega,\mathbf{z}) \in L^+$.
- $\|U(t,\omega)\,\mathbf{z}\| \le \eta\, e^{\beta t}\,\|\mathbf{z}\|$ for every $t \le 0$ and $(\omega,\mathbf{z}) \in L^-$.

We will omit the words "over Ω" when the family (2.1) has ED, since no confusion arises. Let us summarize in the next list of remarks some well-known fundamental properties satisfied by a family of linear Hamiltonian systems which has ED. Detailed proofs and the names of the authors of the results can be found in Chapter 1 of [27].

Remarks 2.2. 1. The ED is unique (in the sense that so are the subbundles L^+ and L^-), and it precludes the existence of globally bounded solutions for any of the systems of the family (2.1). These assertions are also true when the family of linear systems is not of Hamiltonian type: see e.g. Section 1.4.1 of [27].

2. As a matter of fact, the family (2.1) has ED if and only if no one of its systems has a nonzero bounded solution. And the ED of the whole family is equivalent to the ED over $\mathbb{R}$ of each one of its systems.

3. The sections

$$l^{\pm}(\omega) := \{\mathbf{z} \in \mathbb{K}^{2n} \mid (\omega,\mathbf{z}) \in L^{\pm}\} \tag{2.3}$$

are real Lagrange planes. In addition,

$$\begin{aligned} l^{\pm}(\omega) &= \{\mathbf{z}\in\mathbb{K}^{2n} \mid \lim_{t\to\pm\infty} \mathbf{z}(t,\omega,\mathbf{z}_0)=\mathbf{0}\} \\ &= \{\mathbf{z}\in\mathbb{K}^{2n} \mid \sup_{\pm t\in[0,\infty)} \|\mathbf{z}(t,\omega,\mathbf{z}_0)\|<\infty\}\,; \end{aligned} \tag{2.4}$$

and

$$\lim_{t\to\pm\infty} \|\mathbf{z}(t,\omega,\mathbf{z}_0)\| = \infty \quad \text{if } \mathbf{z}_0 \notin l^{\pm}(\omega)\,. \tag{2.5}$$

4. Assume that for all $\omega\in\Omega$, the Lagrange plane $l^+(\omega)$ can be represented by the matrix $\left[\begin{smallmatrix} I_n \\ M^+(\omega)\end{smallmatrix}\right]$. Or, equivalently, that for all $\omega\in\Omega$, the Lagrange plane $l^+(\omega)$ can be represented by a matrix $\left[\begin{smallmatrix} L_1^+(\omega) \\ L_2^+(\omega)\end{smallmatrix}\right]$ with $\det L_1^+(\omega)\neq 0$ (so that $M^+(\omega)=L_2(\omega)\,L_1^{-1}(\omega)$). In this case $M^+\colon\Omega\to\mathbb{S}_n(\mathbb{K})$ is a continuous matrix-valued function, and it is known as one of the *Weyl functions* for (2.1). In this situation, we say that the Weyl function M^+ *globally exists*. In addition, for all $\omega\in\Omega$ the function $t\mapsto M^+(\omega{\cdot}t)$ is a solution of the Riccati equation associated to (2.1), namely

$$M' = -M H_3(\omega{\cdot}t) M - M H_1(\omega{\cdot}t) - H_1^T(\omega{\cdot}t) M + H_2(\omega{\cdot}t)\,. \tag{2.6}$$

We say that M^+ is a *solution along the flow of* (2.6). The other Weyl function is M^-, associated to the subbundle L^-, and it satisfies the same properties (if it exists).

5. Now we do not assume the presence of ED. Let $M(t,\omega,M_0)$ represent the solution of the equation (2.6) corresponding to ω which satisfies $M(0,\omega,M_0)=M_0$. Then the map $(t,\omega,M_0)\mapsto M(t,\omega,M_0)$ defines a continuous skew-product flow on $\Omega\times\mathbb{S}_n(\mathbb{R})$, which is in general local, since the solutions may not be globally defined. In particular, $M(t+s,\omega,M_0)=M(t,\omega{\cdot}s,M(s,\omega,M_0))$ whenever all the elements in the right-hand term are defined.

Definition 2.3. Suppose that the family (2.1) has ED. Then it *satisfies the non-oscillation condition* (or *NC* for short) if the Weyl function M^+ globally exists.

Remark 2.4. It is well-known that ED, NC, and the global existence of M^-, are robust properties, in the sense that each one of them persists under small perturbations of the matrix H of the family (2.1), where small refers to the topology of the uniform convergence in the space of continuous $\mathfrak{sp}(n,\mathbb{K})$-valued functions on Ω. Much more details concerning this persistence, as well as the continuous variation of the respective invariant subbundles and of the Weyl functions can be found in Sections 1.4.5 and 1.4.6 of [27].

Definition 2.5. Let H take values in $\mathfrak{sp}(n,\mathbb{R})$. The family (2.1) of linear Hamiltonian systems is *uniformly weakly disconjugate* (or *UWD* for short) *on* $[0,\infty)$ (resp. *on* $(-\infty,0]$) if there exists $t_0\geq 0$ independent of ω such that for every nonzero solution $\mathbf{z}(t,\omega)=\left[\begin{smallmatrix}\mathbf{z}_1(t,\omega)\\ \mathbf{z}_2(t,\omega)\end{smallmatrix}\right]$ of the systems corresponding to ω with $\mathbf{z}_1(0,\omega)=\mathbf{0}$, there holds $\mathbf{z}_1(t,\omega)\neq\mathbf{0}$ for all $t>t_0$ (resp. $\mathbf{z}_1(t,\omega)\neq\mathbf{0}$ for all $t<-t_0$).

The results summarized in the next remarks can be found in Chapter 5 of [27].

Remarks 2.6. Let us assume that $H_3\geq 0$.

1. The uniform weak disconjugacy (also UWD for short) at $+\infty$ of the family (2.1) is equivalent to the UWD at $-\infty$: see Theorem 5.17 of [27]. We will simply say that the family is UWD.

2. If the family (2.1) is UWD, then there exist *uniform principal solutions at* $\pm\infty$, $\left[\begin{smallmatrix} L_1^\pm(t,\omega) \\ L_2^\pm(t,\omega)\end{smallmatrix}\right]$. They are real $2n\times n$ matrix-valued solutions of (2.1) satisfying the following properties: for all $t\in\mathbb{R}$ and $\omega\in\Omega$, the matrices $L_1^\pm(t,\omega)$ are nonsingular and $\left[\begin{smallmatrix} L_1^\pm(t,\omega) \\ L_2^\pm(t,\omega)\end{smallmatrix}\right]$ represent Lagrange planes; and for all $\omega\in\Omega$,

$$\lim_{\pm t\to\infty}\left(\int_0^t (L_1^\pm)^{-1}(s,\omega)\,H_3(\omega{\cdot}s)\,((L_1^\pm)^T)^{-1}(s,\omega)\,ds\right)^{-1} = 0_n\,.$$

3. If the matrix-valued functions $\left[\begin{smallmatrix} L_1^\pm(t,\omega) \\ L_2^\pm(t,\omega)\end{smallmatrix}\right]$ are uniform principal solutions at $\pm\infty$, then the real matrix-valued functions $N^\pm\colon\Omega\to\mathbb{S}_n(\mathbb{R})\,,\ \omega\mapsto N^\pm(\omega):=L_2^\pm(0,\omega)\,(L_1^\pm(0,\omega))^{-1}$ are unique. They are called *principal functions of* (2.1), and they are solutions along the flow (see Remark 2.2.4) of the Riccati equation (2.6).

Many properties relating the ED of the family (2.1) to its UWD will be used in the proofs of the results of Section 4. All of them can be found in Chapter 5 of [27]. To avoid further interruption in the discussion, we formulate and prove at this point a lemma concerning this relation for a particular type of family (2.1).

Lemma 2.7. *Let H take values in $\mathfrak{sp}(n,\mathbb{R})$, and let us suppose that $H_2>0$ and $H_3>0$. Then,*

(i) *the family of systems* (2.1) *is UWD and has ED, the Weyl functions globally exist and agree with the principal functions, and they satisfy $\mp M^\pm>0$.*

(ii) *If the function $\widetilde M\colon\Omega\to\mathbb{S}_n(\mathbb{R})$ is continuous and a globally defined solution along the flow of* (2.6) *(i.e., if the map $t\mapsto\widetilde M(\omega{\cdot}t)$ is a globally defined solution of the equation for all $\omega\in\Omega$) with $\widetilde M\ge 0$ (resp. with $\widetilde M\le 0$), then $\widetilde M>0$ (resp. $\widetilde M<0$).*

Proof. (i) This assertion is proved by Proposition 5.64(ii) of [27], since the conditions $H_2>0$ and $H_3>0$ guarantee conditions D2 and D2* required in that result: see Remark 5.19 and the comments previous to Proposition 5.64.

(ii) Let us denote $h(\omega,M):=-MH_3(\omega)M-MH_1(\omega)-H_1^T(\omega)M+H_2(\omega)$, and represent by $M(t,\omega,M_0)$ the maximal solution of (2.6) (i.e., of $M'=h(\omega{\cdot}t,M)$) with $M(t,\omega,M_0)=M_0$. Since, by (i), $M^+(\omega)<0<M^-(\omega)$, then the monotonicity properties of the Riccati equation (see Theorem 1.54 of [27])) ensure that $M^+(\omega{\cdot}t)\le M(t,\omega,0_n)\le M^-(\omega{\cdot}t)$ for t in the interval of definition of $M(t,\omega,0_n)$, so that this interval is $\mathbb{R}$ (see e.g. Remark 1.43 of [27]). Since $H_2>0$, we can take $\varepsilon>0$ such that $h(\omega,M(0,\omega,0_n))=h(\omega,0_n)=H_2(\omega)>\varepsilon I_n$. The compactness of Ω allows us to find $t_0>0$ such that $h(\omega{\cdot}t,M(t,\omega,0_n))\ge\varepsilon I_n$ for all $\omega\in\Omega$ and $t\in[-t_0,t_0]$, which ensures that

$$M(-t_0,\omega,0_n)<-\varepsilon\,t_0\,I_n \quad\text{and}\quad M(t_0,\omega,0_n)>\varepsilon\,t_0\,I_n \quad\text{for all } \omega\in\Omega\,. \tag{2.7}$$

Now assume that the function $\widetilde M$ of (ii) satisfies $\widetilde M\ge 0$. The monotonicity yields $\widetilde M(\omega)=M(t_0,\omega{\cdot}(-t_0),\widetilde M(\omega{\cdot}(-t_0)))\ge M(t_0,\omega{\cdot}(-t_0),0_n)>0$, where t_0 satisfies (2.7). The argument is analogous if $\widetilde M\le 0$. □

The last fundamental concept required for the proofs of the main results is that of rotation number with respect to a given σ-ergodic measure. Among the many equivalent definitions for this quantity, we give one which extends that which is possibly the best known in dimension 2. Recall that $U(t,\omega) = \left[\begin{smallmatrix} U_1(t,\omega) & U_3(t,\omega) \\ U_2(t,\omega) & U_4(t,\omega) \end{smallmatrix}\right]$ is the matrix-valued solution of (2.1) with $U(0,\omega) = I_{2n}$. And $\arg\colon \mathbb{C} \to \mathbb{R}$ holds for the continuous branch of the argument of a complex number satisfying $\arg 1 = 0$.

Definition 2.8. Let m_0 be a σ-ergodic measure on Ω. The *rotation number of the family* (2.1) *with respect to* m_0 is the value of

$$\lim_{t\to\infty} \frac{1}{t}\, \arg\det(U_1(t,\omega) - iU_2(t,\omega))\,,$$

for m_0-a.a. $\omega \in \Omega$, which exists, is finite and common.

The proof that this definition is correct can be found in Chapter 2 of [27], where the interested reader will also find many other (equivalent) definitions for the rotation number of different nature as well as an exhaustive description of its properties.

3. Global existence of Weyl functions

Let (Ω, σ) be a real continuous global flow on a compact metric space, and let us denote $\omega{\cdot}t = \sigma(t,\omega)$. Let us consider a continuous matrix-valued function $H\colon \Omega \to \mathfrak{sp}(n,\mathbb{R})$, $H = \left[\begin{smallmatrix} H_1 & H_3 \\ H_2 & H_4 \end{smallmatrix}\right]$, which provides the family of linear Hamiltonian systems systems

$$\mathbf{z}' = H(\omega{\cdot}t)\,\mathbf{z} \tag{3.1}$$

for $\omega \in \Omega$. Given a continuous matrix-valued function $\Delta\colon \Omega \to \mathbb{S}_n(\mathbb{R})$, we consider the perturbed families of Hamiltonian systems

$$\mathbf{z}' = H^\lambda(\omega{\cdot}t)\,\mathbf{z}\,, \quad \text{where } \; H^\lambda(\omega) := \begin{bmatrix} H_1(\omega) & H_3(\omega) + \lambda\Delta(\omega) \\ H_2(\omega) & -H_1^T(\omega) \end{bmatrix} \tag{3.2}$$

for $\omega \in \Omega$. The parameter λ varies in $\mathbb{C}$, and we will use the notation $(3.2)^\lambda$ to refer to a particular system of the family. Obviously, $(3.2)^0$ agrees with (3.1).

We will analyze in this section two different scenarios with $\Delta > 0$ under which, if $\lambda \in \mathbb{C} - \mathbb{R}$, the families $(3.2)^\lambda$ have ED and there exist both Weyl functions, which we will denote by $M^\pm(\omega,\lambda)$. These results will be used in the proof of the main results in Section 4, but have independent interest. In particular, Theorem 3.6 analyzes this question in the absence of the so-called *Atkinson condition* (see (3.5)), which is usually required to guarantee the mentioned properties. It is also important to emphasize that, in the two cases, the Weyl functions will be Herglotz functions on the complex upper and lower half-planes for each fixed $\omega \in \Omega$. As usual, we represent $\mathbb{C}^\pm := \{\lambda \in \mathbb{C} \mid \pm \operatorname{Im}\lambda > 0\}$.

Definition 3.1. A symmetric matrix-valued function M defined on $\mathbb{C}^+$ or $\mathbb{C}^-$ is *Herglotz* if it is holomorphic and $\operatorname{Im} M(\lambda)$ is either positive semidefinite or negative semidefinite on the whole half-plane.

Let us represent by $\mathbf{z}(t,\omega,\mathbf{z}_0) = \left[\begin{smallmatrix} \mathbf{z}_1(t,\omega,\mathbf{z}_0) \\ \mathbf{z}_2(t,\omega,\mathbf{z}_0) \end{smallmatrix}\right]$ the solution of the system (3.1) corresponding to ω which satisfies $\mathbf{z}(0,\omega,\mathbf{z}_0) = \mathbf{z}_0$. The first result (which as a matter of fact is not new: see its proof) is formulated under the next Atkinson-type condition on Δ.

Hypotheses 3.2. $\Delta \ge 0$, and each minimal subset of Ω contains at least one point ω_0 such that

$$\int_{-\infty}^{\infty} \|\Delta(\omega_0{\cdot}t)\,\mathbf{z}_2(t,\omega_0,\mathbf{z}_0)\|^2\, dt > 0 \quad \text{whenever } \mathbf{z}_0 \in \mathbb{C}^{2n} - \{\mathbf{0}\}\,. \tag{3.3}$$

Theorem 3.3. *Suppose that Hypotheses 3.2 hold.*

(i) *If* $\operatorname{Im}\lambda \neq 0$*, then the family* $(3.2)^\lambda$ *has exponential dichotomy.*

(ii) *If* $\operatorname{Im}\lambda \neq 0$*, then there globally exist the Weyl functions* $M^\pm(\omega,\lambda)$*. In addition, the maps* $M^\pm\colon \Omega\times(\mathbb{C}-\mathbb{R})\to\mathbb{S}_n(\mathbb{C})$*,* $(\omega,\lambda)\mapsto M^\pm(\omega,\lambda)$ *are jointly continuous, satisfy* $\pm\operatorname{Im}\lambda\,\operatorname{Im}M^\pm(\omega,\lambda)>0$*, and are holomorphic on* $\mathbb{C}-\mathbb{R}$ *for each* $\omega\in\Omega$ *fixed. In particular, they are Herglotz functions on* $\mathbb{C}^+$ *and* $\mathbb{C}^-$*.*

Proof. In the general case of perturbed Hamiltonian system

$$\mathbf{z}' = \big(H(\omega{\cdot}t)+\lambda J^{-1}\Gamma(\omega{\cdot}t)\big)\,\mathbf{z} \tag{3.4}$$

for a continuous perturbation matrix-valued function $\Gamma\colon\Omega\to\mathbb{S}_{2n}(\mathbb{R})$, all the conclusions of Theorem 3.3 hold under the following general Atkinson condition: $\Gamma\geq 0$, and each minimal subset of Ω contains at least one point ω_0 such that

$$\int_{-\infty}^{\infty}\|\Gamma(\omega_0{\cdot}t)\,\mathbf{z}(t,\omega_0,\mathbf{z}_0)\|^2\,dt>0\quad\text{whenever } \mathbf{z}_0\in\mathbb{C}^{2n}-\{\mathbf{0}\}\,. \tag{3.5}$$

This assertion is originally proved in [21], and a very detailed proof can be found in Theorems 3.8 and 3.9 of [27]. It is also clear that in the case of (3.2), $H^\lambda = H+\lambda J^{-1}\Gamma$ for $\Gamma := \left[\begin{smallmatrix}0_n & 0_n\\ 0_n & \Delta\end{smallmatrix}\right]$, and hence that Hypotheses 3.2 is the general one applied to the particular case. □

Remarks 3.4. 1. It is very easy to check that a function $\mathbf{z}(t,\omega)=\left[\begin{smallmatrix}\mathbf{z}_1(t,\omega)\\ \mathbf{0}\end{smallmatrix}\right]$ solves the system $(3.2)^{\lambda_0}$ corresponding to ω for a $\lambda_0\in\mathbb{C}$ if and only if it solves the system $(3.2)^\lambda$ corresponding to the same ω for all $\lambda\in\mathbb{C}$: both conditions are equivalent to saying that $\mathbf{z}_1'(t,\omega)=H_1(\omega{\cdot}t)\,\mathbf{z}_1(t,\omega)$ and $\mathbf{0}=H_2(\omega{\cdot}t)\,\mathbf{z}_1(t,\omega)$, so that λ plays no role.

2. Let us assume that $\Delta>0$. Then, Δ does not satisfy Hypotheses 3.2 if and only if there exist $\omega\in\Omega$ and $\mathbf{z}_0\in\mathbb{C}^{2n}-\{\mathbf{0}\}$ such that $\mathbf{z}(t,\omega,\mathbf{z}_0)=\left[\begin{smallmatrix}\mathbf{z}_1(t,\omega,\mathbf{z}_0)\\ \mathbf{0}\end{smallmatrix}\right]$ for all $t\in\mathbb{R}$. This assertion follows easily from Lemma 3.6(iv) of [27], which ensures that Δ satisfies Hypotheses 3.2 (or, equivalently, $\Gamma:=\left[\begin{smallmatrix}0_n & 0_n\\ 0_n & \Delta\end{smallmatrix}\right]$ satisfies (3.5)) if and only if

$$\int_{-\infty}^{\infty}\|\Delta(\omega{\cdot}t)\,\mathbf{z}_2(t,\omega,\mathbf{z}_0)\|^2\,dt>0\quad\text{whenever } \mathbf{z}_0\in\mathbb{C}^{2n}-\{\mathbf{0}\}$$

for all $\omega\in\Omega$. (This is for instance the case when $\Delta=I_n$ and $H=\left[\begin{smallmatrix}I_n & 0_n\\ 0_n & I_n\end{smallmatrix}\right]$.) According to the previous remark, $\Delta>0$ does not satisfy Hypotheses 3.2 if and only if there exists a point $\omega\in\Omega$ and a $\lambda_0\in\mathbb{C}$ such that the system $(3.2)^{\lambda_0}$ admits a nontrivial solution $\mathbf{z}(t,\omega)=\left[\begin{smallmatrix}\mathbf{z}_1(t,\omega)\\ \mathbf{0}\end{smallmatrix}\right]$, in which case this function solves the system $(3.2)^\lambda$ for the same ω and all $\lambda\in\mathbb{C}$ (see the previous remark).

3. As a matter of fact, $\Delta>0$ does not satisfy Hypotheses 3.2 if and only if there exist a minimal subset $\mathcal{M}\in\Omega$ such that all the systems $(3.2)^\lambda$ corresponding to $\omega\in\mathcal{M}$ admit a nontrivial solution $\mathbf{z}(t,\omega)=\left[\begin{smallmatrix}\mathbf{z}_1(t,\omega)\\ \mathbf{0}\end{smallmatrix}\right]$ (common for all $\lambda\in\mathbb{C}$).

4. Let $U_{H_1}(t,\omega)$ represent the matrix-valued solution of $\mathbf{z}_1'=H_1(\omega{\cdot}t)\,\mathbf{z}_1$ with $U_{H_1}(0,\omega)=I_n$. Note that $\mathbf{z}(t)=\left[\begin{smallmatrix}\mathbf{z}_1(t)\\ \mathbf{0}\end{smallmatrix}\right]\not\equiv\mathbf{0}$ is a solution of the system (3.1) corresponding to a point $\omega\in\Omega$ if and only $\mathbf{z}_1(0)=\mathbf{z}_1^0\neq\mathbf{0}$ with $H_2(\omega{\cdot}t)\,U_{H_1}(t,\omega)\,\mathbf{z}_1(0)=\mathbf{0}$ for any $t\in\mathbb{R}$, in which case $\mathbf{z}_1(t)=U_{H_1}(t,\omega)\,\mathbf{z}_0^1$. Since $\mathbf{z}_1(t)\neq\mathbf{0}$ for all $t\in\mathbb{R}$, the existence of such a solution ensures that $\det H_2(\omega{\cdot}t)=0$ for all $t\in\mathbb{R}$. By continuity, there must exist a minimal set (contained in the omega limit of ω for the base flow) on which $\det H_2$ vanish identically.

5. Note finally that a continuous and symmetric map $\Gamma\colon \Omega \to \mathbb{M}_{2n\times 2n}(\mathbb{R})$ with $\Gamma > 0$ satisfies (3.5) for all $\omega_0 \in \Omega$, and hence all the conclusions of Theorem 3.3 apply to the family (3.4). This property, which we use later, is ensured by Theorems 3.8 and 3.9 of [27]. (As a matter of fact, it is enough that each minimal subset of Ω contains a point ω with $\Gamma(\omega) > 0$. This is an important difference with the condition $\Delta > 0$ included in Hypotheses 3.5, which we cannot weaken in an analogous way.)

The previous Remarks 3.4.2-4 describe possible situations in which a continuous matrix-valued function $\Delta\colon \Omega \to \mathbb{S}_n(\mathbb{R})$ with $\Delta > 0$ may not satisfy Hypotheses 3.2. The next result will also prove the occurrence of ED and the global existence of Weyl functions for λ outside the real line, under a different condition. The point $\lambda_0 \in \mathbb{C}$ appearing in its hypotheses can of course be real.

Hypotheses 3.5. $\Delta > 0$, and there exists $\lambda_0 \in \mathbb{C}$ such that the family $(3.2)^{\lambda_0}$ has exponential dichotomy.

Theorem 3.6. *Suppose that Hypotheses 3.5 hold.*

(i) *If* $\operatorname{Im}\lambda \neq 0$, *then the family* $(3.2)^{\lambda}$ *has exponential dichotomy.*

(ii) *If* $\operatorname{Im}\lambda \neq 0$, *there globally exist the Weyl functions* $M^{\pm}(\omega,\lambda)$. *In addition, the maps* $M^{\pm}\colon \Omega \times (\mathbb{C}-\mathbb{R}) \to \mathbb{S}_n(\mathbb{C})$, $(\omega,\lambda) \mapsto M^{\pm}(\omega,\lambda)$ *are jointly continuous, satisfy* $\pm\operatorname{Im}\lambda\,\operatorname{Im}M^{\pm}(\omega,\lambda) \geq 0$, *and are holomorphic on* $\mathbb{C}-\mathbb{R}$ *for each* $\omega \in \Omega$ *fixed. In particular, they are Herglotz functions on* $\mathbb{C}^+$ *and* $\mathbb{C}^-$.

(iii) *If Hypotheses 3.2 do not hold, there exists a minimal subset* $\mathcal{M} \subseteq \Omega$ *such that either* $\det M^+(\omega,\lambda) = 0$ *for all* $\omega \in \mathcal{M}$ *and all* $\lambda \in \mathbb{C}-\mathbb{R}$ *or* $\det M^-(\omega,\lambda) = 0$ *for all* $\omega \in \mathcal{M}$ *and all* $\lambda \in \mathbb{C}-\mathbb{R}$.

Proof. The arguments that we will use adapt those of the proof of the result corresponding to the Atkinson condition (3.5) (see again Theorem 3.8 of [27]).

(i) We fix $\omega \in \Omega$ and $\lambda_0 \in \mathbb{C}-\mathbb{R}$, and represent $\|\mathbf{z}\|_{\Delta_t} = (\mathbf{z}^*\Delta(\omega{\cdot}t)\,\mathbf{z})^{1/2}$. The main step of the proof shows that the system of the family $(3.2)^{\lambda_0}$ corresponding to our choice of ω does not admit a nonzero bounded solution. We define the functional $\mathcal{L}_\omega^{\lambda_0}$ as

$$(\mathcal{L}_\omega^{\lambda_0}\mathbf{z})(t) = J\mathbf{z}'(t) - JH^{\lambda_0}(\omega{\cdot}t)\,\mathbf{z}(t)\,,$$

and observe that, for any solution $\mathbf{z} = \left[\begin{smallmatrix}\mathbf{z}_1\\ \mathbf{z}_2\end{smallmatrix}\right]$ of the system $(3.2)^{\lambda_0}$ corresponding to ω, we have $\mathcal{L}_\omega^{\lambda_0}\mathbf{z} \equiv \mathbf{0}$ and hence

$$\begin{aligned} 0 &= \int_a^b \big(\mathbf{z}^*(t)\,(\mathcal{L}_\omega^{\lambda_0}\mathbf{z})(t) - (\mathcal{L}_\omega^{\lambda_0}\mathbf{z})^*(t)\,\mathbf{z}(t)\big)\,dt \\ &= \mathbf{z}^*(t)J\mathbf{z}(t)\big|_{t=a}^{t=b} - 2\,i\operatorname{Im}\lambda_0 \int_a^b \|\mathbf{z}_2(t)\|_{\Delta_t}^2\,dt \end{aligned} \tag{3.6}$$

whenever $a < b$. Let us assume for contradiction that there exists a bounded solution $\mathbf{z}(t,\omega,\mathbf{z}_0) = \left[\begin{smallmatrix}\mathbf{z}_1(t,\omega,\mathbf{z}_0)\\ \mathbf{z}_2(t,\omega,\mathbf{z}_0)\end{smallmatrix}\right]$ of $(3.2)^{\lambda_0}$. Then (3.6) ensures that

$$\int_{\mathbb{R}} \|\mathbf{z}_2(t,\omega,\mathbf{z}_0)\|_{\Delta_t}^2\,dt < \infty\,,$$

which provides an increasing sequence $(t_m) \uparrow \infty$ such that

$$\int_{t_m}^{t_m+1} \|\mathbf{z}_2(t,\omega,\mathbf{z}_0)\|_{\Delta_t}^2\,dt < \frac{1}{m}$$

for every $m \in \mathbb{N}$. The compactness of Ω and the boundedness of $(\widetilde{\mathbf{z}}(t_m))$ provide a subsequence (t_j) and points $\widetilde{\omega} \in \Omega$ and $\widetilde{\mathbf{z}}_0 \in \mathbb{C}^{2n}$ such that $\widetilde{\omega} = \lim_{j\to\infty} \omega{\cdot}t_j$ and $\widetilde{\mathbf{z}}_0 = \lim_{j\to\infty} \mathbf{z}(t_j, \omega, \mathbf{z}_0)$. Consequently,

$$\mathbf{z}(t, \widetilde{\omega}, \widetilde{\mathbf{z}}_0) = \lim_{j\to\infty} \mathbf{z}(t, \omega{\cdot}t_j, \mathbf{z}(t_j, \omega, \mathbf{z}_0))\,.$$

Hence, since

$$\frac{1}{j} > \int_{t_j}^{t_j+1} \|\mathbf{z}_2(t,\omega,\mathbf{z}_0)\|^2_{\Delta_t}\, dt = \int_0^1 \|\mathbf{z}_2(t,\omega{\cdot}t_j,\mathbf{z}(t_j,\omega,\mathbf{z}_0))\|^2_{\Delta_t}\, dt\,,$$

we find that

$$\int_0^1 \|\mathbf{z}_2(t,\widetilde{\omega},\widetilde{\mathbf{z}}_0)\|^2_{\Delta_t}\, dt = 0\,,$$

which, since $\Delta > 0$, ensures that $\mathbf{z}_2(0, \widetilde{\omega}, \widetilde{\mathbf{z}}_0) = \mathbf{0}$. In other words,

$$\lim_{j\to\infty} \mathbf{z}_2(t_j, \omega, \mathbf{z}_0) = \mathbf{0}\,.$$

A symmetric argument provides a sequence $(s_j) \downarrow -\infty$ such that

$$\lim_{j\to\infty} \mathbf{z}_2(s_j, \omega, \mathbf{z}_0) = \mathbf{0}\,.$$

Therefore, applying (3.6) to each interval $[s_j, t_j]$ and taking limits as $j \to \infty$ yields

$$\int_{-\infty}^{\infty} \|\mathbf{z}_2(t,\omega,\mathbf{z}_0)\|^2_{\Delta_t}\, dt = 0\,,$$

and since $\delta > 0$ it follows that $\mathbf{z}_2(t, \omega, \mathbf{z}_0) \equiv \mathbf{0}$. This means that $\mathbf{z}(t, \omega, \mathbf{z}_0) = \left[\begin{smallmatrix} \mathbf{z}_1(t,\omega,\mathbf{z}_0) \\ \mathbf{0} \end{smallmatrix}\right]$ is a bounded solution of the system $(3.2)^{\lambda_0}$ corresponding to ω. But it is immediate to check that it also solves the system the system $(3.2)^{\lambda}$ corresponding to this ω for all $\lambda \in \mathbb{C}$, including $\lambda = \lambda_0$. The contradiction has been reached: according to Remark 2.2.1, the existence of this nontrivial bounded solution precludes the exponential dichotomy of $(3.2)^{\lambda_0}$, assumed from the beginning.

(ii) We take $\lambda \in \mathbb{C}-\mathbb{R}$, so that $(3.2)^{\lambda}$ has ED. Let $L^{\pm}_{\lambda}$ be the invariant subbundles appearing in Definition 2.1, and let $l^{\pm}(\omega, \lambda)$ be the corresponding sections, given by (2.3). We also take $\omega \in \Omega$, and assume for contradiction that there exists $\mathbf{z}^0 = \left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2^0 \end{smallmatrix}\right] \in l^+(\omega, \lambda)$. Applying (3.6) to the solution $\mathbf{z}(t, \omega, \mathbf{z}^0)$ on intervals $[0, t]$ for $t > 0$, and having in mind that $\lim_{t\to\infty} \mathbf{z}(t, \omega, \mathbf{z}^0) = \mathbf{0}$ (see (2.4)), we obtain $\int_0^\infty \|\mathbf{z}_2(t,\omega,\mathbf{z}_0)\|^2_{\Delta_t}\, dt = 0$. This ensures that $\mathbf{z}_2(t, \omega, \mathbf{z}^0) = \mathbf{0}$ for any $t \geq 0$. In particular, $\mathbf{z}_2^0 = \mathbf{0}$, so that $\mathbf{z}^0 = \mathbf{0}$. Hence $l^+(\omega, \lambda)$ contains no nontrivial vectors of the form $\left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2 \end{smallmatrix}\right]$, and thus it can be represented by $\left[\begin{smallmatrix} I_n \\ M^+(\omega,\lambda) \end{smallmatrix}\right]$, where $M^{\pm}(\omega, \lambda)$ is symmetric. An analogous argument shows the global existence of $M^-(\omega, \lambda)$. The continuity of the map $M^+(\omega, \lambda)$ on $\Omega \times (\mathbb{C}-\mathbb{R})$ follows for instance from the Sacker and Sell perturbation theorem (see Theorem 6 of [42] or Theorem 1.95 of [27]). The holomorphic character of $\lambda \mapsto M^{\pm}(\omega, \lambda)$ outside the real axis can be proved repeating the argument of the proof of Theorem 3.9 of [27].

It remains to prove that $\pm \operatorname{Im}\lambda \operatorname{Im} M^{\pm}(\omega, \lambda) \geq 0$. To this end, we consider the new auxiliary perturbed systems $\mathbf{z}' = H^{\lambda}_k(\omega{\cdot}t)\,\mathbf{z}$ with $H^{\lambda}_k = H + \lambda J^{-1}\Gamma_k$ for $\Gamma_k = \left[\begin{smallmatrix} (1/k)\, I_n & 0_n \\ 0_n & \Delta \end{smallmatrix}\right]$. Since $\Gamma_k > 0$ for $k \geq 1$, it satisfies the general Atkinson condition and the conclusions of Theorem 3.3 hold (see Remark 3.4.5); thus, if $\lambda \in \mathbb{C} - \mathbb{R}$, then there exist the corresponding Weyl functions $M^{\pm}_k(\omega, \lambda)$ and they satisfy $\pm\,\mathrm{Im}\,\lambda\, M_k^\pm(\omega,\lambda)>0$. Fix $\lambda\notin\mathbb{R}$, and note that $\lim_{k\to\infty} H_k^\lambda = H^\lambda$ uniformly on Ω. Therefore, the Sacker and Sell perturbation theorem (see again Theorem 6 of [42] or Theorem 1.95 of [27]) ensures that $\lim_{k\to\infty} M_k^\pm(\omega,\lambda)=M^\pm(\omega,\lambda)$. Consequently, $\pm\,\mathrm{Im}\,\lambda\,\mathrm{Im}\, M^\pm(\omega,\lambda)\ge 0$, which completes the proof of (ii).

(iii) Since condition (3.3) does not hold, there exists a minimal subset $\mathcal{M}\subseteq\Omega$ and a nontrivial solution of the form $\mathbf{z}(t,\omega)=\left[\begin{smallmatrix}\mathbf{z}_1(t,\omega)\\ \mathbf{0}\end{smallmatrix}\right]$ of the systems corresponding to $\omega\in\mathcal{M}$ of the families $(3.2)^\lambda$ for all $\lambda\in\mathbb{C}$: see Remark 3.4.3. Let us fix $\widetilde\lambda\in\mathbb{C}-\mathbb{R}$, so that the functions $M^\pm(\omega,\lambda)$ globally exist. Let us also fix $\widetilde\omega\in\mathcal{M}$.

If $\mathbf{z}(0,\widetilde\omega)=\left[\begin{smallmatrix}\mathbf{z}_1(0,\widetilde\omega)\\ \mathbf{0}\end{smallmatrix}\right]$ belongs to $l^+(\widetilde\omega,\widetilde\lambda)$, then $\mathbf{z}(t,\widetilde\omega)=\left[\begin{smallmatrix}\mathbf{z}_1(t,\widetilde\omega)\\ \mathbf{0}\end{smallmatrix}\right]$ belongs to $l^+(\widetilde\omega{\cdot}t,\lambda)$ for all $t\in\mathbb{R}$. Since $l^+(\widetilde\omega{\cdot}t,\widetilde\lambda)$ can be represented by $\left[\begin{smallmatrix}I_n\\ M^+(\widetilde\omega\cdot t,\widetilde\lambda)\end{smallmatrix}\right]$, we have $\det M^+(\widetilde\omega{\cdot}t,\widetilde\lambda)=0$. The continuity of M^+ and the minimality of $\mathcal{M}$ ensure that $\det M^+(\omega,\widetilde\lambda)=0$ for all $\omega\in\mathcal{M}$.

Note now that $\mathbf{z}(0,\widetilde\omega)=\left[\begin{smallmatrix}\mathbf{z}_1(0,\widetilde\omega)\\ \mathbf{0}\end{smallmatrix}\right]$ belongs to $l^+(\widetilde\omega,\lambda)$ for all $\lambda\in\mathbb{C}-\mathbb{R}$, as we deduce from Remark 3.4.1 and from the characterization (2.4) of the Lagrange plane. Therefore the previous argument can be repeated in order to show that $\det M^+(\omega,\lambda)=0$ for all $\omega\in\mathcal{M}$ and all $\lambda\in\mathbb{C}-\mathbb{R}$.

In the remaining cases, $\mathbf{z}(0,\widetilde\omega)=\mathbf{z}_0^+ + \mathbf{z}_0^-$ with $\mathbf{z}_0^\pm\in l^\pm(\widetilde\omega,\widetilde\lambda)$ and $\mathbf{z}_0^-\neq\mathbf{0}$, and it follows from (2.5) that

$$\lim_{t\to\infty}\|\mathbf{z}(t,\widetilde\omega)\|=\infty\,. \tag{3.7}$$

We denote $\mathbf{z}^\pm(t,\widetilde\omega)=\mathbf{z}(t,\widetilde\omega,\mathbf{z}_0^\pm)$ and observe that $\mathbf{z}(t,\widetilde\omega)=\mathbf{z}^+(t,\widetilde\omega)+\mathbf{z}^-(t,\widetilde\omega)$ and $(\widetilde\omega{\cdot}t,\mathbf{z}^\pm(t,\widetilde\omega))\in L^\pm_{\widetilde\lambda}$ for all $t\in\mathbb{R}$. Now we take $\omega\in\mathcal{M}$ and choose $(t_m)\uparrow\infty$ with $\lim_{m\to\infty}\widetilde\omega{\cdot}t_m=\omega$ and such that there exists $\mathbf{z}^*:=\lim_{m\to\infty}\mathbf{z}(t_m,\widetilde\omega)/\,\|\mathbf{z}(t_m,\widetilde\omega)\|$. It follows from (2.4) and (3.7) that $\lim_{m\to\infty}\mathbf{z}^+(t_m,\widetilde\omega)/\,\|\mathbf{z}(t_m,\widetilde\omega)\|=\mathbf{0}$, so that $\lim_{m\to\infty}\mathbf{z}^-(t_m,\widetilde\omega)/\,\|\mathbf{z}(t_m,\widetilde\omega)\|=\lim_{m\to\infty}\mathbf{z}(t_m,\widetilde\omega)/\,\|\mathbf{z}(t_m,\widetilde\omega)\|=\mathbf{z}^*$. The closed character of $L^-_{\widetilde\lambda}$ and the fact that $\mathbf{z}^-(t_m,\widetilde\omega)/\,\|\mathbf{z}(t_m,\widetilde\omega)\|\in l^-(\widetilde\omega{\cdot}t_m,\widetilde\lambda)$ ensure that $\mathbf{z}^*\in l^-(\omega,\widetilde\lambda)$. Note that $\mathbf{z}^*\in l^-(\omega,\lambda)$ is the initial data of a solution of the form $\mathbf{z}^*(t,\widetilde\omega)=\left[\begin{smallmatrix}\mathbf{z}_1^*(t,\widetilde\omega)\\ \mathbf{0}\end{smallmatrix}\right]$ with initial data in $l^+(\widetilde\omega,\lambda)$ for all $\lambda\in\mathbb{C}-\mathbb{R}$: see again Remark 3.4.1. Therefore, the argument used in the first situation shows that $\det M^-(\omega,\lambda)=0$ for all $\omega\in\mathcal{M}$ and all $\lambda\in\mathbb{C}-\mathbb{R}$. This completes the proof. □

The statement of the previous theorem and the proof of its point (i) prove the next result.

Corollary 3.7. *Suppose that the continuous matrix-valued function $\Delta\colon\Omega\to\mathbb{S}_n(\mathbb{R})$ takes positive definite values. Then, there are two dynamical possibilities for the families $(3.2)^\lambda$:*

- O1. *There exist $\lambda_0\in\mathbb{C}$ such that the family $(3.2)^{\lambda_0}$ has exponential dichotomy. In this case the families $(3.2)^\lambda$ have exponential dichotomy for (at least) all $\lambda\in\mathbb{C}-\mathbb{R}$, and the Weyl functions $M^\pm(\omega,\lambda)$ globally exist for all $\lambda\in\mathbb{C}-\mathbb{R}$ and are Herglotz functions.*
- O2. *The family $(3.2)^\lambda$ does not have exponential dichotomy for any $\lambda\in\mathbb{C}$. Equivalently, there exists a point $\omega\in\Omega$ and a $\lambda_0\in\mathbb{C}$ such that the system $(3.2)^{\lambda_0}$ corresponding to ω admits a nonzero bounded solution of the form $\mathbf{z}(t,\omega)=\left[\begin{smallmatrix}\mathbf{z}_1(t,\omega)\\ \mathbf{0}\end{smallmatrix}\right]$, in which case this function solves the system $(3.2)^\lambda$ corresponding to ω for all $\lambda\in\mathbb{C}$.*

Note that situation O2 is extremely non-persistent. For instance, O1 holds in the following cases:

- When Δ satisfies the Atkinson Hypotheses 3.2, as Theorem 3.3 ensures.
- When $\det H_2$ does not vanish identically on any minimal subset $\mathcal{M} \subset \Omega$: Remark 3.4.4 ensures that in this case Δ satisfies the Atkinson Hypotheses 3.2.
- If the n-dimensional family of systems $\mathbf{z}_1' = H_1(\omega{\cdot}t)\,\mathbf{z}_1$ has exponential dichotomy, since any nonzero solution $\left[\begin{smallmatrix}\mathbf{z}_1(t,\omega)\\ \mathbf{0}\end{smallmatrix}\right]$ of (3.2) provides a nonzero solution of $\mathbf{z}_1(t,\omega)$ of $\mathbf{z}_1' = H_1(\omega{\cdot}t)\,\mathbf{z}_1$ which cannot be bounded (see Remark 2.2.1).

We conclude this section with another consequence of Theorem 3.6 which concerns other type of perturbed systems, namely

$$\mathbf{z}' = \widetilde{H}^\lambda(\omega{\cdot}t)\,\mathbf{z}\,, \quad \text{where } \widetilde{H}^\lambda(\omega) := \begin{bmatrix} H_1(\omega{\cdot}t) & H_3(\omega) \\ H_2(\omega) + \lambda\Delta(\omega) & -H_1^T(\omega) \end{bmatrix}. \tag{3.8}$$

Corollary 3.8. *Suppose that the continuous matrix-valued function $\Delta\colon \Omega \to \mathbb{S}_n(\mathbb{R})$ takes positive definite values. Then, there are two dynamical possibilities for the families $(3.8)^\lambda$:*

O1*. *There exist $\lambda_0 \in \mathbb{C}$ such that the family $(3.8)^{\lambda_0}$ has exponential dichotomy. In this case the families $(3.2)^\lambda$ have exponential dichotomy for (at least) all $\lambda \in \mathbb{C} - \mathbb{R}$.*

O2*. *The family $(3.8)^\lambda$ does not have exponential dichotomy for any $\lambda \in \mathbb{C}$. Equivalently, there exists a point $\omega \in \Omega$ and a $\lambda_0 \in \mathbb{C}$ such that the system $(3.8)^{\lambda_0}$ corresponding to ω admits a nonzero bounded solution of the form $\mathbf{z}(t,\omega) = \left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t,\omega)\end{smallmatrix}\right]$, in which case this function solves the system $(3.8)^\lambda$ corresponding to ω for all $\lambda \in \mathbb{C}$.*

Proof. It is easy to check that the change of variables $\mathbf{w} = \left[\begin{smallmatrix}0_n & I_n\\ I_n & 0_n\end{smallmatrix}\right]\mathbf{z}$ takes (3.8) to

$$\mathbf{w}' = \begin{bmatrix} -H_1^T(\omega{\cdot}t) & H_2(\omega{\cdot}t) + \lambda\Delta(\omega{\cdot}t) \\ H_3(\omega{\cdot}t) & H_1(\omega{\cdot}t) \end{bmatrix}\mathbf{w}\,, \tag{3.9}$$

which is in one of the situations described in Corollary 3.7. Obviously a nonzero bounded solution exists for one of the systems of $(3.9)^\lambda$ if and only a nonzero bounded solution exists for one of the systems of $(3.8)^\lambda$. This fact allows us to deduce from Remark 2.2.2 that the ED holds or not simultaneously for $(3.9)^\lambda$ and $(3.8)^\lambda$. Therefore, the assertions follow from Corollary 3.7. □

Remarks 3.9. 1. Note that a family of the type (3.8) arises when dealing with the n-dimensional Schrödinger family $\mathbf{x}' + G(\omega{\cdot}t)\,\mathbf{x} = \lambda\Delta(\omega{\cdot}t)$, by taking $\mathbf{z} = \left[\begin{smallmatrix}\mathbf{x}\\ \mathbf{x}'\end{smallmatrix}\right]$. It is known (and very easy to check) that, in this case, the perturbation matrix $\Gamma = \left[\begin{smallmatrix}\Delta & 0_n\\ 0_n & 0_n\end{smallmatrix}\right]$ satisfies the general Atkinson condition (3.5), so that the statements of Theorem 3.3 hold in this case. In particular, the Schrödinger case is in situation O1*. But clearly the situation that we consider in Corollary 3.8 is much more general.

2. A linear Hamiltonian system admitting a nontrivial bounded solution of the form $\mathbf{z}(t) = \left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t)\end{smallmatrix}\right]$ on a positive of negative half-line is called *abnormal at* $+\infty$ or *at* $-\infty$. Note that, in situation O2*, each one of the families $(3.8)^\lambda$ has an abnormal system both at $+\infty$ and at $-\infty$. The reader is referred to [36, 37, 46, 47, 48, 10, 24] and references therein, for an analysis of abnormal linear Hamiltonian systems.

4. Exponential dichotomy and nonoscillation condition for parametric families

As in the previous section, (Ω,σ) is a real continuous global flow on a compact metric space, and $\omega{\cdot}t=\sigma(t,\omega)$. Given continuous functions $H\colon\Omega\to\mathfrak{sp}(n,\mathbb{R})$ with $H=\left[\begin{smallmatrix} H_1 & H_3\\ H_2 & H_4\end{smallmatrix}\right]$ and $\Delta\colon\Omega\to\mathbb{S}_n(\mathbb{R})$, we consider the families of linear Hamiltonian systems

$$\mathbf{z}'=H(\omega{\cdot}t)\,\mathbf{z} \tag{4.1}$$

and

$$\mathbf{z}'=H^\lambda(\omega{\cdot}t)\,\mathbf{z}\,,\quad\text{where}\;\; H^\lambda(\omega):=\begin{bmatrix} H_1(\omega) & H_3(\omega)\\ H_2(\omega)-\lambda\Delta(\omega) & -H_1^T(\omega)\end{bmatrix} \tag{4.2}$$

for $\omega\in\Omega$. The parameter λ may vary in $\mathbb{C}$, although our results will refer to real values of λ. We will use the notation $(4.2)^\lambda$ to refer to the family corresponding to a particular value of λ. Note that $(4.2)^0$ and (4.1) coincide.

The concepts of ED and NC appearing in the next set of conditions, under which the results of this section will be obtained, are given in Definitions 2.1 and 2.3.

Hypotheses 4.1. $H_3\ge 0$, $\Delta>0$, and the family (4.1) has exponential dichotomy and it satisfies the nonoscillation condition.

In the formulation of the results, new sets of families of linear Hamiltonian system will play a role. The first one is

$$\mathbf{z}'=H_\varepsilon^\lambda(\omega{\cdot}t)\,\mathbf{z}\,,\quad\text{where}\;\; H_\varepsilon^\lambda(\omega):=\begin{bmatrix} H_1(\omega) & H_3(\omega)+\varepsilon I_n\\ H_2(\omega)-\lambda\Delta(\omega) & -H_1^T(\omega)\end{bmatrix}\mathbf{z} \tag{4.3}$$

for $\varepsilon\in\mathbb{R}$. We will use the notation $(4.3)_\varepsilon^\lambda$ to refer to the family corresponding to particular values of λ and ε. Note that $(4.3)_0^\lambda$ and $(4.2)^\lambda$ agree. We will also work with

$$\mathbf{w}'=\begin{bmatrix} -H_1^T(\omega{\cdot}t) & H_2(\omega{\cdot}t)-\lambda\Delta(\omega{\cdot}t)\\ H_3(\omega{\cdot}t) & H_1(\omega{\cdot}t)\end{bmatrix}\mathbf{w} \tag{4.4}$$

for $\lambda\in\mathbb{C}$, as well as with

$$\mathbf{w}'=\begin{bmatrix} -H_1^T(\omega{\cdot}t) & H_2(\omega{\cdot}t)-\lambda\Delta(\omega{\cdot}t)\\ H_3(\omega{\cdot}t)+\varepsilon I_n & H_1(\omega{\cdot}t)\end{bmatrix}\mathbf{w} \tag{4.5}$$

for $\lambda\in\mathbb{C}$ and $\varepsilon\in\mathbb{R}$. We represent by $(4.4)^\lambda$ and $(4.5)_\varepsilon^\lambda$ the families corresponding to a particular (real or complex) value of λ. Again, $(4.5)_0^\lambda$ agrees with $(4.4)^\lambda$. In all the cases, we will often substitute λ by α if the parameter is real

Before explaining the goals of this section, we will derive some facts from the relation of $(4.5)_\varepsilon^\lambda$ and $(4.4)^\lambda$ with $(4.3)_\varepsilon^\lambda$ and $(4.2)^\lambda$, and we will also establish some notation which we will use below.

Remarks 4.2. 1. As in the proof of Corollary 3.8, it is easy to check that the change of variables $\mathbf{w}=\left[\begin{smallmatrix} 0_n & I_n\\ I_n & 0_n\end{smallmatrix}\right]\mathbf{z}$ takes $(4.2)^\lambda$ to $(4.4)^\lambda$ and $(4.3)_\varepsilon^\lambda$ to $(4.5)_\varepsilon^\lambda$; hence, according to Remark 2.2.2, the ED of $(4.3)_\varepsilon^\lambda$ is equivalent to the ED of $(4.5)_\varepsilon^\lambda$. Moreover, it follows from (2.3) that, in the case of ED, $\left[\begin{smallmatrix} z_1\\ z_2\end{smallmatrix}\right]\in l_\varepsilon^\pm(\omega,\lambda)$ if and only if $\left[\begin{smallmatrix} z_2\\ z_1\end{smallmatrix}\right]\in\widetilde{l}_\varepsilon^\pm(\omega,\lambda)$, where $l_\varepsilon^\pm(\omega,\lambda)$ and $l_\varepsilon^\pm(\omega,\lambda)$ are the Lagrange planes defined by (2.3) from the families $(4.3)_\varepsilon^\lambda$ and $(4.5)_\varepsilon^\lambda$. And of course, the same happens with $(4.2)^\lambda$ and $(4.4)^\lambda$.

2. In the case of ED and of global existence of the Weyl functions (just one or both of them) for $(4.3)_\varepsilon^\lambda$ (or for $(4.5)_\varepsilon^\lambda$) for $\lambda\in\mathbb{C}$ and $\varepsilon\in\mathbb{R}$, we will represent them

by $M^{\pm}_{\varepsilon}(\omega,\lambda)$ (or by $\widetilde{M}^{\pm}_{\varepsilon}(\omega,\lambda)$). In particular, $M^{\pm}_0(\omega,\lambda)$ (or $\widetilde{M}^{\pm}_0(\omega,\lambda)$) represent the Weyl functions of $(4.2)^{\lambda}$ (or of $(4.4)^{\lambda}$), for which we will omit the subscript: $M^{\pm}(\omega,\lambda)$ (or $\widetilde{M}^{\pm}(\omega,\lambda)$).

3. Similarly, in the case of UWD of the family $(4.3)^{\alpha}_{\varepsilon}$ (or of $(4.5)^{\alpha}_{\varepsilon}$) for $\alpha\in\mathbb{R}$ and $\varepsilon\in\mathbb{R}$, we will represent the corresponding principal functions by $N^{\pm}_{\varepsilon}(\omega,\alpha)$ (or by $\widetilde{N}^{\pm}_{\varepsilon}(\omega,\alpha)$); and we will denote $N^{\pm}(\omega,\alpha):=N^{\pm}_0(\omega,\alpha)$ (and $\widetilde{N}^{\pm}(\omega,\alpha):=\widetilde{N}^{\pm}_0(\omega,\alpha)$).

4. Suppose that both $M^{+}_{\varepsilon}(\omega,\lambda)$ and $\widetilde{M}^{+}_{\varepsilon}(\omega,\lambda)$ exist. The relation between the solutions of $(4.3)^{\lambda}_{\varepsilon}$ and $(4.5)^{\lambda}_{\varepsilon}$ show that $\mathbf{z}=\left[\begin{smallmatrix}\mathbf{z}_1\\ \mathbf{z}_2\end{smallmatrix}\right]$ belongs to the Lagrange plane given by $\left[\begin{smallmatrix}I_n\\ M^{+}_{\varepsilon}(\omega,\alpha)\end{smallmatrix}\right]$ if and only if $\mathbf{w}=\left[\begin{smallmatrix}\mathbf{z}_2\\ \mathbf{z}_1\end{smallmatrix}\right]$ belongs to that given by $\left[\begin{smallmatrix}M^{+}_{\varepsilon}(\omega,\alpha)\\ I_n\end{smallmatrix}\right]$, which agrees with that given by $\left[\begin{smallmatrix}I_n\\ (M^{+}_{\varepsilon})^{-1}(\omega,\alpha)\end{smallmatrix}\right]$. This fact combined with (2.4) guarantees that $\widetilde{M}^{+}_{\varepsilon}(\omega,\lambda)=(M^{+}_{\varepsilon})^{-1}(\omega,\lambda)$. The same property holds of course in the case of $M^{-}_{\varepsilon}(\omega,\lambda)$ and $\widetilde{M}^{-}_{\varepsilon}(\omega,\lambda)$.

5. For the same reason, suppose that $(4.5)^{\lambda}_{\varepsilon}$ has ED, and that $\widetilde{M}^{+}_{\varepsilon}(\omega,\lambda)$ exists and is nonsingular for all $\omega\in\Omega$. Then $M^{+}_{\varepsilon}(\omega,\lambda)$ exists and $M^{+}_{\varepsilon}(\omega,\lambda)=(\widetilde{M}^{+}_{\varepsilon})^{-1}(\omega,\lambda)$. And the analogous property holds in the case of $\widetilde{M}^{-}_{\varepsilon}(\omega,\lambda)$ and $M^{-}_{\varepsilon}(\omega,\lambda)$.

The main result of this section, Theorem 4.4, improves the following result, which is part of Theorem 7.31 of [27], and was previously analyzed in [26]. As a matter of fact, the result of [27] is formulated in the case that Ω is minimal and assuming that a certain condition D2 does not hold, but an identical proof works for the statement we give now. (Note that the minimality of the flow ensures that any σ-ergodic measure on Ω has full topological support, which is a condition that Theorem 4.4 requires.)

Theorem 4.3. *Suppose that Hypotheses 4.1 hold. Let us define*

$$\begin{aligned}\mathcal{I}_0:=\{0\}\cup\{\alpha_0\in\mathbb{R}\,|\,(4.2)^{\alpha}\ &\textit{has ED and satisfies NC}\\ &\textit{for all }\alpha\in[0,\alpha_0)\textit{ or }\alpha\in(\alpha_0,0]\}\}\,.\end{aligned}\tag{4.6}$$

Then,

(i) *$\mathcal{I}_0$ is an open interval containing 0, and $M^{+}(\omega,\alpha_1)<M^{+}(\omega,\alpha_2)$ for every $\omega\in\Omega$ and for every pair of elements $\alpha_1<\alpha_2$ of $\mathcal{I}_0$.*

(ii) *There exists a nonincreasing and lower semicontinuous extended-real function $\rho\colon\mathcal{I}_0\to(0,\infty]$ such that $(4.3)^{\alpha}_{\varepsilon}$ has ED and is UWD for $\alpha\in\mathcal{I}_0$ if and only if $\varepsilon\in(0,\rho(\alpha))$. In particular, for these values of ε, there exist the Weyl functions $M^{\pm}_{\varepsilon}(\omega,\alpha)$.*

Theorem 4.4. *Suppose that Hypotheses 4.1 hold, and that there exists a σ-ergodic measure m_0 on Ω with full topological support. Let us define*

$$\mathcal{I}:=\{\alpha\in\mathbb{R}\,|\,(4.2)^{\alpha}\ \textit{has ED and satisfies NC}\}\,,\tag{4.7}$$

and $\mathcal{I}_0$ by (4.6). Then,

(i) *there exists $\alpha^*\in(0,\infty]$ such that $\mathcal{I}=(-\infty,\alpha^*)=\mathcal{I}_0$, and $M^{+}(\omega,\alpha_1)<M^{+}(\omega,\alpha_2)$ for every $\omega\in\Omega$ and for every pair of elements $\alpha_1<\alpha_2$ of $\mathcal{I}$.*

(ii) *The map $\rho\colon\mathcal{I}\to(0,\infty]$ defined in Theorem 4.3(ii) is continuous and nonincreasing.*

(iii) *In addition, $\rho(\alpha)=\infty$ whenever $H_2-\alpha\Delta>0$, and ρ is striclty decreasing at the points at which it takes real values (if they exist).*

(iv) *If $\alpha^*<\infty$, then the family $(4.2)^{\alpha^*}$ does not have ED.*

Before proving it, we will state and prove an auxiliary result, which does not require the existence of a σ-ergodic measure with full support.

Theorem 4.5. *Suppose that Hypotheses 4.1 hold, and let α_0 be a real value with $H_2 - \alpha_0\Delta > 0$. Then, if $\alpha \le \alpha_0$, the family $(4.4)^\alpha$ is UWD and has ED, and the Weyl functions $\widetilde{M}^\pm(\omega,\alpha)$ globally exist; and, in addition*

$$\widetilde{M}^+(\omega,\alpha_1) \le \widetilde{M}^+(\omega,\alpha_2) < 0 \le \widetilde{M}^-(\omega,\alpha_2) \le \widetilde{M}^-(\omega,\alpha_1)$$

if $\alpha_2 < \alpha_1 \le \alpha_0$.

Consequently, if $\alpha \le \alpha_0$, the family $(4.2)^\alpha$ has ED, and the Weyl function $M^+(\omega,\alpha)$ globally exists; and, in addition

$$M^+(\omega,\alpha_2) \le M^+(\omega,\alpha_1) < 0$$

if $\alpha_2 < \alpha_1 \le \alpha_0$.

Proof. Throughout the proof, we will use the notation established in Remarks 4.2.2&3.

Proposition 5.27 and Remark 5.19 of [27] show that the family $(4.4)^\alpha$ is UWD for $\alpha < \alpha_0$, and Proposition 5.51(i) of [27] ensures that the corresponding principal functions $\widetilde{N}^\pm(\omega,\alpha)$ satisfy

$$\widetilde{N}^+(\omega,\alpha_1) \le \widetilde{N}^+(\omega,\alpha_2) \le \widetilde{N}^-(\omega,\alpha_2) \le \widetilde{N}^-(\omega,\alpha_1) \quad \text{if } \alpha_2 < \alpha_1 \le \alpha_0\,. \tag{4.8}$$

The proof of the theorem will be complete once we check that the family $(4.4)^\alpha$ satisfies $\widetilde{N}^+(\omega,\alpha) < 0 \le \widetilde{N}^-(\omega,\alpha)$ if $\alpha < \alpha_0$: in these conditions the principal functions determine Theorem 5.58 of [27] ensures that, if $\alpha < \alpha_0$, then the family $(4.4)^\alpha$ has ED and admits Weyl functions with $\widetilde{M}^\pm(\omega,\alpha) = \widetilde{N}^\pm(\omega,\alpha)$; and these facts, (4.8), and Remarks 4.2.1&5 prove all the assertions.

From this point, and for the safe of clarity, we divide the proof in three steps.

First step. We will start by proving the next properties:

1. For any $\lambda \in \mathbb{C} - \mathbb{R}$ there globally exist the Weyl functions $\widetilde{M}^\pm(\omega,\lambda)$. In addition, the maps $M^\pm\colon \Omega \times (\mathbb{C} - \mathbb{R}) \to \mathbb{S}_n(\mathbb{C})$, $(\omega,\lambda) \mapsto M^\pm(\omega,\lambda)$ are jointly continuous, and they are holomorphic on $\mathbb{C} - \mathbb{R}$ for each $\omega \in \Omega$ fixed. In particular, they are Herglotz functions on $\mathbb{C}^+$ and $\mathbb{C}^-$.
2. Moreover, $\mp \operatorname{Re} \widetilde{M}^\pm(\omega,\lambda) \ge 0$ and $\mp \operatorname{Im} \widetilde{M}^\pm(\omega,\lambda) \ge 0$ whenever $\operatorname{Re}\lambda \le \alpha_0$ and $\operatorname{Im}\lambda > 0$.

Let us take $\varepsilon > 0$ and $\alpha \in (-\infty,\alpha_0]$. Since $H_2 - \alpha\Delta > 0$ and $H_3 + \varepsilon I_n > 0$, Lemma 2.7(i) ensures that the family $(4.3)^\alpha_\varepsilon$ has ED and that the corresponding (real) Weyl functions $M^\pm_\varepsilon(\omega,\alpha)$ exist and satisfy

$$\mp M^\pm_\varepsilon(\omega,\alpha) > 0 \quad \text{for all } \omega \in \Omega \text{ if } \varepsilon > 0 \text{ and } \alpha \le \alpha_0\,. \tag{4.9}$$

Therefore, if $\varepsilon > 0$ and $\alpha \le \alpha_0$, then the family $(4.5)^\alpha_\varepsilon$ has ED and there exist the Weyl functions $\widetilde{M}^\pm_\varepsilon(\omega,\alpha)$ for $(4.2)^\alpha_\varepsilon$, with $\mp\widetilde{M}^\pm_\varepsilon(\omega,\alpha) = \mp(M^\pm_\varepsilon)^{-1}(\omega,\alpha)$: see Remarks 4.2.1&5. In particular, it follows from (4.9) that

$$\mp \widetilde{M}^\pm_\varepsilon(\omega,\alpha) > 0 \quad \text{for all } \omega \in \Omega \text{ if } \varepsilon > 0 \text{ and } \alpha \le \alpha_0\,. \tag{4.10}$$

Hypotheses 4.1 and Remark 4.2.1 ensure that the family $(4.5)^0_0$ has ED. Hence there exists ε_0 such that $(4.5)^0_\varepsilon$ has ED for $\varepsilon \in [0,\varepsilon_0)$: see Remark 2.4. We deduce from Theorem 3.6(i)&(ii) the next properties: if $\varepsilon \in [0,\varepsilon_0)$, $\alpha \in \mathbb{R}$ and

$\beta > 0$, then the family $(4.5)_\varepsilon^{\alpha+i\beta}$ has ED and there globally exist the Weyl functions $\widetilde{M}_\varepsilon^\pm(\omega,\alpha+i\beta)$. Note that $\widetilde{M}_0^\pm(\omega,\alpha+i\beta)=\widetilde{M}^\pm(\omega,\alpha+i\beta)$. The information provided in Theorem 3.6(ii) concerning continuity and analyticity completes the proof of property **1**. In addition, also according to Theorem 3.6(ii),

$$\mp\,\mathrm{Im}\,\widetilde{M}_\varepsilon^\pm(\omega,\alpha+i\beta)\ge 0 \quad \text{if } \varepsilon\in[0,\varepsilon_0),\ \alpha\in\mathbb{R} \text{ and } \beta>0\,. \tag{4.11}$$

Moreover, we can ensure that

$$\mp\,\mathrm{Re}\,\widetilde{M}_\varepsilon^\pm(\omega,\alpha+i\beta)>0 \quad \text{for all } \omega\in\Omega \ \text{ if } \varepsilon\in(0,\varepsilon_0),\ \alpha\le\alpha_0 \text{ and } \beta>0\,. \tag{4.12}$$

In order to prove this last assertion, we fix $\varepsilon\in(0,\varepsilon_0)$ and $\alpha\le\alpha_0$, and use the Sacker and Sell perturbation theorem (see Theorem 6 of [42] or Theorem 1.95 of [27]) to ensure that

$$\lim_{\beta\to 0^+}\widetilde{M}_\varepsilon^\pm(\omega,\alpha+i\beta)=\widetilde{M}_\varepsilon^\pm(\omega,\alpha) \quad \text{uniformly in } \omega\in\Omega\,. \tag{4.13}$$

Therefore, it follows from (4.10) that there exists $\beta_0=\beta(\varepsilon,\alpha)$ such that

$$\mp\,\mathrm{Re}\,\widetilde{M}_\varepsilon^\pm(\omega,\alpha+i\beta)>0 \quad \text{for all } \omega\in\Omega \ \text{ if } \beta\in[0,\beta_0)\,.$$

Let us work now with $\widetilde{M}^-$, assuming for contradiction that the value

$$\beta^*=\beta^*(\varepsilon,\alpha):=\sup\{\beta_0>0\,|\ \mathrm{Re}\,\widetilde{M}_\varepsilon^-(\omega,\alpha+i\beta)>0 \text{ for all } \omega\in\Omega \text{ if } \beta\in[0,\beta_0)\}$$

is finite. We denote $R^\pm(\omega):=\mathrm{Re}\,\widetilde{M}_\varepsilon^\pm(\omega,\alpha+i\beta^*)$ and $I^\pm(\omega):=\mathrm{Im}\,\widetilde{M}_\varepsilon^\pm(\omega,\alpha+i\beta^*)$. A straightforward computation from the Riccati equation (see (2.6)) associated to $(4.5)_\varepsilon^{\alpha+i\beta^*}$ shows that, for all $\omega\in\Omega$, the maps $t\mapsto R^\pm(\omega{\cdot}t)$ are solutions of the Riccati equation associated to the (real) Hamiltonian system

$$\mathbf{u}'=\begin{bmatrix} -H_1^T+\beta\Delta I^\pm & H_2-\alpha\Delta \\ I^\pm(H_2-\alpha\Delta)\,I^\pm+H_3+\varepsilon I_n & H_1-\beta I^\pm\Delta \end{bmatrix}\mathbf{u}$$

(where H_1, H_2, H_3, Δ and $I^\pm$ have argument $\omega{\cdot}t$). We know also that they are globally defined and that $R^-(\omega)\ge 0$. Since $H_2-\alpha\Delta>0$ and $H_2-\alpha\Delta+H_3+\varepsilon I_n>0$, Lemma 2.7(ii) shows that $R^-(\omega)>0$, which contradicts the definition of β^*. So, (4.12) is proved for $\widetilde{M}^-$, and the proof for $\widetilde{M}^+$ is analogous.

Finally, we can also deduce from the Sacker and Sell perturbation theorem that

$$\begin{gathered}\lim_{\varepsilon\to 0^+}\widetilde{M}_\varepsilon^\pm(\omega,\lambda)=\widetilde{M}^\pm(\omega,\lambda) \quad \text{uniformly} \\ \text{in the compact sets of } \Omega\times\mathbb{C}^+ \text{ and } \Omega\times\mathbb{C}^-\,,\end{gathered} \tag{4.14}$$

which together with (4.11) and (4.12) proves property **2**. This completes the first step.

Second step. We fix $\omega\in\Omega$ and prove the next assertions.

3. Let $\widetilde{\alpha}_0>0$ satisfy $H_2-\widetilde{\alpha}_0\Delta>0$. Then $\lim_{\beta\to 0^+}\mathrm{Im}\,\widetilde{M}^-(\omega,\alpha+i\beta)=0_n$ uniformly on the compact subsets of $(-\infty,\widetilde{\alpha}_0)$.

4. In addition, there exist the limits $\widetilde{F}^\pm(\omega,\alpha):=\lim_{\beta\to 0^+}\widetilde{M}^\pm(\omega,\alpha+i\beta)$ for all $\alpha\le\alpha_0$, and they are real matrices with $\mp\widetilde{F}^\pm(\omega,\alpha)\ge 0$. Moreover, the matrix-valued functions $t\mapsto\widetilde{F}^\pm(\omega{\cdot}t,\alpha)$ are two globally defined solutions of the Riccati equation associated to $(4.4)^\alpha$.

Note that, in the first step, nothing precludes us from substituting α_0 by a slightly greater $\widetilde{\alpha}_0$ for which $H_2-\widetilde{\alpha}_0\Delta>0$: Properties **2** and (4.9) are true for this $\widetilde{\alpha}_0$. We will use both of them. Let ε_0 be the real number defined at the beginning of the first step. As seen in (4.11), the holomorphic maps $\mathbb{C}^+\to\mathbb{S}_n(\mathbb{C})$, $\lambda\mapsto\widetilde{M}^-_\varepsilon(\omega,\lambda)$ are Herglotz for $\varepsilon\in[0,\varepsilon_0)$ (see Definition 3.1). Therefore, Theorem 4.6(ii) (see below) provides the representation

$$\widetilde{M}^-_\varepsilon(\omega,\lambda)=L_\varepsilon+K_\varepsilon\,\lambda+\int_{\mathbb{R}}\left(\frac{1}{t-\lambda}-\frac{t}{t^2+1}\right)dP_\varepsilon(t)$$

for $\operatorname{Im}\lambda>0$ and $\varepsilon\in[0,\varepsilon_0)$. In the case $\varepsilon=0$ we rewrite this as

$$\widetilde{M}^-(\omega,\lambda)=L+K\,\lambda+\int_{\mathbb{R}}\left(\frac{1}{t-\lambda}-\frac{t}{t^2+1}\right)dP(t)\,.$$

Let us take $\varepsilon\in(0,\varepsilon_0)$. Theorem 4.6(iii) can be combined with the property (4.13) and with the real character of $\widetilde{M}^-_\varepsilon(\omega,\alpha)$ for $\alpha\le\widetilde{\alpha}_0$ (see (4.9)) in order to see that

$$\frac{1}{2}\left(P_\varepsilon\{\alpha_1\}+P_\varepsilon\{\alpha_2\}\right)+\int_{(\alpha_1,\alpha_2)}dP_\varepsilon(t)=\frac{1}{\pi}\int_{\alpha_1}^{\alpha_2}\operatorname{Im}\widetilde{M}^-_\varepsilon(\omega{\cdot}d,\alpha)\,d\alpha=0_n$$

whenever $\alpha_1<\alpha_2\le\widetilde{\alpha}_0$. This ensures that

$$\int_{(-\infty,\widetilde{\alpha}_0)}dP_\varepsilon(t)=0_n\quad\text{if }\varepsilon\in(0,\varepsilon_0)\,. \tag{4.15}$$

Now we take a sequence $(\varepsilon_m)\downarrow 0$ and recall that $\lim_{m\to\infty}\widetilde{M}^-_{\varepsilon_m}(\omega,\lambda)=\widetilde{M}^-(\omega,\lambda)$ uniformly on the compact sets of $\mathbb{C}^+$ (see (4.14)). Therefore the sequence (dP_{ε_m}) converges to dP in the weak* sense (see Theorem 3.15 of [27]), which together with (4.15) allows us to check that

$$\int_{(-\infty,\widetilde{\alpha}_0)}dP(t)=0_n\,. \tag{4.16}$$

Let us take $\delta>0$ and $\alpha_1<\widetilde{\alpha}_0-\delta$ and denote $\mathcal{C}^\delta_{\alpha_1}:=\{\lambda\in\mathbb{C}\mid\ \operatorname{Re}\lambda\in[\alpha_1,\widetilde{\alpha}_0-\delta]$ and $\operatorname{Im}\lambda\in(0,1]\}$. Note that there exist constants $c_1>0$ and $c_2>c_1$ such that

$$c_1\le\frac{|t-\lambda|^2}{t^2+1}\le c_2\quad\text{for}\ \ t\in[\widetilde{\alpha}_0,\infty)\ \ \text{and}\ \ \lambda\in\mathcal{C}^\delta_{\alpha_1}\,, \tag{4.17}$$

since the function is continuous, takes strictly positive values (due to $|t-\lambda|^2\ge\delta^2$), and $\lim_{t\to\infty}|t-\lambda|^2/(t^2+1)=1$. Then, if $\lambda=\alpha+i\beta\in\mathcal{C}^\delta_{\alpha_1}$,

$$\begin{aligned}\frac{1}{\beta}\,\operatorname{Im}\widetilde{M}^-(\omega,\alpha+i\beta)&=K+\int_{\mathbb{R}}\frac{1}{|t-\lambda|^2}\,dP(t)\\&=K+\int_{[\alpha_0,\infty)}\frac{1}{|t-\lambda|^2}\,dP(t)\le K+\frac{1}{c_1}\int_{[\alpha_0,\infty)}\frac{1}{t^2+1}\,dP(t)\\&\le K+\frac{1}{c_1}\int_{\mathbb{R}}\frac{1}{t^2+1}\,dP(t)\le K+\frac{1}{c_1}\,\operatorname{Im}\widetilde{M}^-(\omega,i)\,.\end{aligned}$$

Here we have used Theorem (4.6)(ii) at the first and last steps, and (4.16) and (4.17) at the second and third steps. This and property **2** yield

$$0_n\le\operatorname{Im}\widetilde{M}^-(\omega,\alpha+i\beta)\le\beta\left(K+\frac{1}{c_1}\,\operatorname{Im}\widetilde{M}^-(\omega,i)\right)$$

if $\alpha\in[\alpha_1,\widetilde{\alpha}_0-\delta]$ and $\beta\in(0,1]$. Property **3** follows easily from here.

In turn, property **3** guarantees that whenever a sequence (λ_m) in $\mathbb{C}^+$ converges to $\alpha \in (-\infty, \widetilde{\alpha}_0)$, it is $\lim_{m\to\infty} \operatorname{Im} \widetilde{M}^-(\omega, \lambda_m) = 0_n$. This fact allows us to apply the Schwarz reflection principle (see e.g. [38], Theorem 11.14) in order to ensure that $\widetilde{M}^-(\omega, \lambda)$ admits a holomorphic extension to $\mathbb{C} - (\alpha_0, \infty)$ (which is contained in $\mathbb{C} - [\widetilde{\alpha}_0, \infty)$) with null imaginary part for $\lambda = \alpha \in (-\infty, \alpha_0]$. We call $\widetilde{F}^-(\omega, \alpha)$ to the restriction of this extension to $(-\infty, \alpha_0]$. Note that this process can be performed for all $\omega \in \Omega$. In particular,

$$\widetilde{F}(\omega{\cdot}t, \alpha) = \lim_{\beta\to 0^+} \widetilde{M}^-(\omega{\cdot}t, \alpha + i\beta)$$

for all $t \in \mathbb{R}$. Since $t \mapsto \widetilde{M}^-(\omega{\cdot}t, \alpha + i\beta)$ solves the Riccati equation associated to $(4.4)^{\alpha+i\beta}$, we conclude that $t \mapsto \widetilde{F}^-(\omega{\cdot}t, \alpha)$ solves the Riccati equation associated to $(4.4)^{\alpha}$. Finally, it follows from **2** that $\widetilde{F}^-(\omega, \alpha) \ge 0$. Hence, property **4** is proved.

The proofs of **3** and **4** are analogous in the case of $\widetilde{M}^+$: the second step is complete.

Third step. Property **4** combined with Theorem 5.48(iii) of [27] ensures that

$$\widetilde{N}^+(\omega, \alpha) \le \widetilde{F}^+(\omega, \alpha) \le 0 \le \widetilde{F}^-(\omega, \alpha) \le \widetilde{N}^-(\omega, \alpha) \quad \text{if } \alpha \in (-\infty, \alpha_0] \tag{4.18}$$

for all $\omega \in \Omega$, where $\widetilde{N}^\pm(\omega, \alpha)$ are the principal functions for $(4.4)^\alpha$, whose existence has been guaranteed at the beginning of the proof. This fact will allow us to prove the next assertion.

5. $\widetilde{N}^+(\omega, \alpha) < 0 \le \widetilde{N}^-(\omega, \alpha)$ for all $\omega \in \Omega$ if $\alpha < \alpha_0$.

Note that, as explained before the first step, this property completes the proof. Note also that the second inequality is already proved: see (4.18).

So we must just prove that $\widetilde{N}^+(\omega, \alpha) < 0$. We will use below this immediate consequence of Theorem 4.3(i):

$$\text{there exists } \alpha_1 > 0 \text{ such that } [-\alpha_1, \alpha_1] \subset \mathcal{I}\,. \tag{4.19}$$

Let us proceed by contradiction, assuming that there exists $\widetilde{\alpha} \le \alpha_0$ such that "$\widetilde{N}^+(\omega, \widetilde{\alpha}) < 0$" is false. Then, since (4.8) holds, there exists $\widetilde{\omega} \in \Omega$ and a vector $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} \ne \mathbf{0}$ (which we fix from now on), such that $\mathbf{v}^T \widetilde{N}^+(\widetilde{\omega}, \widetilde{\alpha})\, \mathbf{v} = 0$. It also follows from (4.8) and (4.18) that

$$\mathbf{v}^T \widetilde{N}^+(\widetilde{\omega}, \alpha)\, \mathbf{v} = 0 \quad \text{if } \alpha \le \widetilde{\alpha}\,. \tag{4.20}$$

Let us define the holomorphic function $h\colon \mathbb{C}^+ \to \mathbb{C}$ by

$$h(\alpha + i\beta) := \mathbf{v}^T \widetilde{M}^+(\widetilde{\omega}, \alpha + i\beta)\, \mathbf{v}\,.$$

As seen in the proof of property **3**, there exists a holomorphic extension $\widetilde{h}$ of h to $\mathbb{C} - [\widetilde{\alpha}, \infty)$. Note also that (4.18) and (4.20) yield

$$\widetilde{h}(\alpha) = \lim_{\beta\to 0^+} h(\alpha + i\beta) = \mathbf{v}^T \widetilde{F}^+(\widetilde{\omega}, \alpha)\, \mathbf{v} = 0 \quad \text{for all } \alpha \le \widetilde{\alpha}\,.$$

The principle of isolated zeroes ensures then that

$$\widetilde{h}(\alpha + i\beta) = \mathbf{v}^T \widetilde{M}^+(\widetilde{\omega}, \alpha + i\beta)\, \mathbf{v} = 0 \quad \text{if } \alpha \in \mathbb{R} \text{ and } \beta > 0\,.$$

Therefore $\mathbf{v}^T \operatorname{Re} \widetilde{M}^+(\widetilde{\omega}, \alpha + i\beta)\, \mathbf{v} = 0$ and $\mathbf{v}^T \operatorname{Im} \widetilde{M}^+(\widetilde{\omega}, \alpha + i\beta)\, \mathbf{v} = 0$ for $\alpha \in \mathbb{R}$ and $\beta > 0$. These equalities combined with property **2** ensure that, if $\alpha \le \widetilde{\alpha}$ and $\beta > 0$, then $\operatorname{Re} \widetilde{M}^+(\widetilde{\omega}, \alpha + i\beta)\, \mathbf{v} = \mathbf{0}$ and $\operatorname{Im} \widetilde{M}^+(\widetilde{\omega}, \alpha + i\beta)\, \mathbf{v} = \mathbf{0}$. In other words,

$\widetilde{M}^+(\widetilde{\omega},\alpha+i\beta)\,\mathbf{v}=\mathbf{0}$ if $\alpha\le\widetilde{\alpha}$ and $\beta>0$. A new application of the principle of isolated zeroes, now to components of the holomorphic vector function $\widetilde{M}^+(\widetilde{\omega},\alpha+i\beta)\,\mathbf{v}$ defined on the upper complex half-plane, shows that $\widetilde{M}^+(\widetilde{\omega},\alpha+i\beta)\,\mathbf{v}=\mathbf{0}$ if $\alpha\in\mathbb{R}$ and $\beta>0$. Now we use property **3** in order to deduce from Theorem 4.6(i) the existence of a point $\alpha_2\in[-\alpha_1,\alpha_1]$ (where α_1 satisfies (4.19)) such that there exists

$$\widetilde{F}^+(\widetilde{\omega},\alpha_2):=\lim_{\beta\to0^+}\widetilde{M}^+(\widetilde{\omega},\alpha_2+i\beta)\,, \tag{4.21}$$

so that $\widetilde{F}^+(\widetilde{\omega},\alpha_2)\,\mathbf{v}=\mathbf{0}$. The Sacker and Sell perturbation theorem (see once more Theorem 6 of [42] or Theorem 1.95 of [27]) combined with (4.21) ensure that the Lagrange plane $\widetilde{l}^+(\widetilde{\omega},\alpha_2)$ of $(4.4)^{\alpha_2}$ (see (2.3)) can be represented by $\left[\begin{smallmatrix} I_n\\ \widetilde{F}^+(\widetilde{\omega},\alpha_2)\end{smallmatrix}\right]$. And we already know that it can be also represented by $\left[\begin{smallmatrix} M^+(\widetilde{\omega},\alpha_2)\\ I_n\end{smallmatrix}\right]$: see Remark 4.2.1. This means that $\widetilde{F}^+(\widetilde{\omega},\alpha_2)$ is a nonsingular matrix (it agrees with $(M^+)^{-1}(\widetilde{\omega},\alpha_2)$), which contradicts the equality $\widetilde{F}^+(\widetilde{\omega},\alpha_2)\,\mathbf{v}=\mathbf{0}$. We have arrived to the sought-for contradiction, and hence the proof is complete. □

Let us formulate at this point the result on Herglotz matrix-valued functions which we have used in the proof of Theorem 4.5. It can be found in [29] and [17].

Theorem 4.6. *Let $G\colon\mathbb{C}^+\to\mathbb{S}_n(\mathbb{C})$ be a Herglotz function, with $\operatorname{Im}G\ge0$. Then,*

(i) *for Lebesgue a.e. $\alpha\in\mathbb{R}$ there exists the nontangential limit from the upper half-plane $\lim_{\lambda\searrow\alpha}G(\lambda)$.*

(ii) *There exist real symmetric matrices L and K and a real matrix-valued function $P(t)$ defined for $t\in\mathbb{R}$, which is symmetric, nondecreasing and right-continuous, such that the* Nevalinna–Riesz–Herglotz representation

$$G(\lambda)=L+K\,\lambda+\int_{\mathbb{R}}\left(\frac{1}{t-\lambda}-\frac{t}{t^2+1}\right)dP(t) \tag{4.22}$$

holds for $\lambda\in\mathbb{C}^+$, with

$$L=\operatorname{Re}G(i)\qquad\textit{and}\qquad K=\lim_{\beta\to\infty}\frac{1}{i\beta}\,G(i\beta)\ge0\,.$$

(iii) *Let us represent $P\{\alpha\}=P(\alpha^+)-P(\alpha^-)=P(\alpha)-\lim_{\mu\to\alpha^-}P(\mu)$ for $\alpha\in\mathbb{R}$. The* Stieltjes inversion formula

$$\frac{1}{2}\left(P\{\alpha_1\}+P\{\alpha_2\}\right)+\int_{(\alpha_1,\alpha_2)}dP(t)=\frac{1}{\pi}\lim_{\beta\to0^+}\int_{\alpha_1}^{\alpha_2}\operatorname{Im}G(\alpha+i\beta)\,d\alpha$$

holds. In addition,

$$P\{\alpha\}=\lim_{\beta\to0^+}\beta\,\operatorname{Im}G(\alpha+i\beta)=-i\lim_{\beta\to0^+}\beta\,G(\alpha+i\beta)\,,$$
$$0=\lim_{\beta\to0^+}\beta\,\operatorname{Re}G(\alpha+i\beta)\,.$$

In particular, the matrix-valued measure dP in representation (4.22) *is uniquely determined.*

Now we can finally prove our main result.

Proof of Theorem 4.4. The notation established in Remarks 4.2 will be used in this proof.

(i) It is obvious that $\mathcal{I}_0 \subseteq \mathcal{I}$, where $\mathcal{I}_0$ and $\mathcal{I}$ are respectively defined by (4.6) and (4.7). We will prove that

$$(-\infty, 0] \subset \mathcal{I}\,. \tag{4.23}$$

Let us first explain why this proves (i). If there exists $\alpha_* \in I$ with $\alpha_* > 0$, we can replace (4.1) by $(4.4)^{\alpha_*}$ in Hypothesis 4.1 in order to conclude that $(-\infty, \alpha_*] \subset \mathcal{I}$. This ensures that $\mathcal{I}$ is either a negative half-line (containing $(-\infty, 0]$) or the whole $\mathbb{R}$. It follows trivially that $\mathcal{I}$ agrees with $\mathcal{I}_0$, so that Theorem 4.3(i) completes the proof of (i).

So, proving (4.23) is our goal. Let us take $\alpha_0 \in \mathbb{R}$ as in the statement of Theorem 4.5, so that the systems $(4.2)^\alpha$ with $\alpha \in (-\infty, \alpha_0]$ have ED and satisfy NC. This ensures that $(-\infty, \alpha_0] \subseteq \mathcal{I}$. We assume that $\alpha_0 < 0$ (otherwise there is nothing to prove). Note that we must just prove that the family $(4.2)^\alpha$ has ED and satisfies NC for $\alpha \in (\alpha_0, 0)$.

We will first prove the assertion concerning ED. Let us fix $\alpha_1 \in (\alpha_0, 0)$. The robustness of the ED and NC (see Remark 2.4) and Theorem 4.3(ii) allow us to choose $\varepsilon_0 < 0$ close enough to 0 as to guarantee these two conditions: the families $(4.3)_\varepsilon^{\alpha_0}$ have ED and satisfy NC for all $\varepsilon \in [\varepsilon_0, 0]$; and if the point $(0, \varepsilon_2)$ belongs to the line $\mathcal{R}$ (in the $(\alpha, \varepsilon)-$plane) which joins $(\alpha_0, \varepsilon_0)$ with $(\alpha_1, 0)$ (so that $\varepsilon_2 > 0$), then the family $(4.3)_{\varepsilon_2}^0$ has ED and is UWD (or, using the words of Theorem 4.3(ii), $\varepsilon_2 \in (0, \rho(0))$). Note that the points of $\mathcal{R}$ are $(\alpha_1 + \gamma, \gamma\,(\varepsilon_2/(-\alpha_1)))$ for $\gamma := \alpha - \alpha_1$, and that the families $(4.3)_\varepsilon^\alpha$ for points $(\alpha, \varepsilon) \in \mathcal{R}$ can be rewritten as

$$\mathbf{z}' = \begin{bmatrix} H_1(\omega{\cdot}t) & H_3(\omega{\cdot}t) + \gamma(-\varepsilon_2/\alpha_1)\, I_n \\ H_2(\omega{\cdot}t) - \alpha_1\Delta(\omega{\cdot}t) - \gamma\Delta(\omega{\cdot}t) & -H_1^T(\omega{\cdot}t) \end{bmatrix} \mathbf{z}\,;$$

that is, as

$$\mathbf{z}' = \big(H^{\alpha_1}(\omega{\cdot}t) + \gamma J^{-1}\Gamma(\omega{\cdot}t)\big)\,\mathbf{z} \tag{4.24}$$

for $\Gamma := \left[\begin{smallmatrix} \Delta & 0_n \\ 0_n & (-\varepsilon_2/\alpha_1)\, I_n \end{smallmatrix}\right]$. Since $\Gamma > 0$, it satisfies the Atkinson condition (3.5) for all $\omega_0 \in \Omega$ (see Remark 3.4.5). In addition, this family has ED for the positive point $\gamma_1 := -\alpha_1$ (since in this case we have the family $(4.3)_{\varepsilon_2}^0$), so that the families corresponding to values of γ close enough to γ_1 also have ED. This fact allow us to apply the results of [22] (see also Theorem 3.50 of [27]) in order to deduce that the rotation number with respect to m_0 is constant for all the families (4.24) corresponding to the values of γ in an open interval centered in γ_1. As a matter of fact, it is 0 at γ_1, since the family $(4.3)_{\varepsilon_2}^0$ satisfies $H_3 + \varepsilon_2 I_n > 0$ and is UWD: see Theorem 5.67 of [27]. The rotation number is also 0 at the negative point $\gamma_0 = \alpha_0 - \alpha_1$. To check this assertion note that for this value of γ, the family agrees with $(4.3)_{\varepsilon_0}^{\alpha_0}$, and that the families $(4.3)_\varepsilon^{\alpha_0}$ have ED for all $\varepsilon \in [\varepsilon_0, 0]$; deduce that the rotation number with respect to m_0 is the same for all these families (applying for instance Theorems 2.28 and 2.25 of [27]); and note that the rotation number of the family $(4.3)_0^{\alpha_0}$, which satisfies NC with $H_3 \geq 0$, is 0 (for instance, having in mind the global existence of the corresponding Weyl function M^+ and applying Propositions 5.9 and 5.65 of [27]). In addition, the rotation number increases as γ increases (see Proposition 2.33 of [27]), Therefore, it is 0 for $\gamma \in [\gamma_0, \gamma_1]$, which combined with the required condition $\operatorname{Supp} m_0 = \Omega$ (see again Theorem 3.50 of [27]) ensures that the families corresponding to (γ_0, γ_1) have ED. This includes the family corresponding to $\gamma = 0$, which is $(4.2)^{\alpha_1}$. Our assertion concerning the ED is proved.

Let us now prove that $(4.2)^\alpha$ also satisfies the NC (i.e., that $M^+(\omega,\alpha)$ globally exists) for $\alpha\in[\alpha_0,0]$, which will complete the proof of (4.23) and hence of (i). We define

$$\widetilde{\mathcal{I}}:=\{\widetilde\alpha\in[\alpha_0,0]\,|\,(4.2)^\alpha \text{ has ED and satisfies NC for all } \alpha\in(-\infty,\widetilde\alpha]\}\,.$$

Note that $\widetilde{\mathcal{I}}$ is a nonempty and open subset of $[\alpha_0,0]$, since $(-\infty,\alpha_0]\subset\mathcal{I}$ (see Remark 2.4); and hence that $\alpha_2:=\sup\widetilde{\mathcal{I}}>\alpha_0$. The goal is to prove that $(4.2)^{\alpha_2}$ satisfies the NC: if so, $\alpha_2\in\widetilde{\mathcal{I}}$, which ensures that $\alpha_2=0$ and hence that $(-\infty,0]\subset\mathcal{I}$. We take a strictly increasing sequence $(\widetilde\alpha_m)$ in $[\alpha_0,\alpha_2)$ with limit α_2, and take $\varepsilon_0>0$ such that, if $\varepsilon\in[0,\varepsilon_0]$, then: the families $(4.3)^\alpha_\varepsilon$ have ED for $\alpha\in[\widetilde\alpha_1,0]$; and there globally exist $M^+_\varepsilon(\omega,0)$. Since $H_3+\varepsilon_0 I_n>0$, we deduce from Remark 5.22 and Theorem 5.17 of [27] that the family $(4.3)^0_{\varepsilon_0}$ is UWD, and from Proposition 5.51 and Remark 5.22 of [27] that all the families $(4.3)^\alpha_\varepsilon$ for $\alpha\in[\widetilde\alpha_1,0]$ and $\varepsilon\in(0,\varepsilon_0]$ are also UWD, with $N^+_\varepsilon(\omega,\alpha)=M^+_\varepsilon(\omega,\alpha)$, and with

$$M^+_\varepsilon(\omega,\widetilde\alpha_m)\le M^+_\varepsilon(\omega,\widetilde\alpha_{m+1})\le M^+_\varepsilon(\omega,\alpha_2)\le M^+_{\varepsilon_0}(\omega,\alpha_2)$$

for all $\omega\in\Omega$. On the other hand, the Sacker and Sell perturbation theorem (see e.g. Theorem 1.95 of [27]) ensures that $M^+(\omega,\alpha)=\lim_{\varepsilon\to0^+}M^+_\varepsilon(\omega,\alpha)$ for all $\alpha\in[\widetilde\lambda_1,\lambda_3)$, so that

$$M^+(\omega,\widetilde\alpha_m)\le M^+(\omega,\widetilde\alpha_{m+1})\le M^+_{\varepsilon_0}(\omega,\alpha_2)\,.$$

Therefore, there exists $F^+(\omega,\alpha_2):=\lim_{m\to\infty}M^+(\omega,\widetilde\alpha_m)$ for all $\omega\in\Omega$, which ensures that the Lagrange plane represented by $\left[\begin{smallmatrix}I_n\\ F^+(\omega,\alpha_2)\end{smallmatrix}\right]$ is the limit in the Lagrangian manifold of those given by $\left[\begin{smallmatrix}I_n\\ M^+(\omega,\widetilde\alpha_m)\end{smallmatrix}\right]$ (see e.g. Proposition 1.25 of [27]); that is, of the sequence $(l^+(\omega,\widetilde\alpha_m))$. The Sacker and Sell perturbation theorem ensures that $\left[\begin{smallmatrix}I_n\\ F^+(\omega,\alpha_2)\end{smallmatrix}\right]$ represents $l^+(\omega,\alpha_2)$, so that $M^+(\omega,\alpha_2)$ globally exists (and agrees with $F^+(\omega,\alpha_2)$). This completes the proof of (i).

(ii) According to Theorem 4.3, the function ρ is lower semicontinuous and nonincreasing, so that it is continuous from the right. Let us take $\alpha_0\in\mathcal{I}$ and a sequence $(\alpha_m)\uparrow\alpha_0$, and call $\rho_0:=\lim_{m\to\infty}\rho(\alpha_m)$ (with $\rho_0\le\infty$). We know that $\rho(\alpha_0)\le\rho_0$, and our goal is to prove that they are equal. Or, in other words, that the family $(4.3)^\varepsilon_{\alpha_0}$ has ED and is UWD for $\varepsilon\in(0,\rho_0)$.

The UWD is deduced by applying Theorem 5.61 of [27] to the families $(4.3)^\alpha_\varepsilon$ for a fixed $\varepsilon\in(0,\rho_0)$ and α varying in $\mathbb{R}$. Let $N^\pm_\varepsilon(\omega,\alpha_0)$ be the corresponding principal functions. Proposition 5.51 of [27] shows that

$$N^+_{\varepsilon_1}(\omega,\alpha_0)\le N^+_{\varepsilon_2}(\omega,\alpha_0)\le N^-_{\varepsilon_2}(\omega,\alpha_0)\le N^-_{\varepsilon_1}(\omega,\alpha_0)\tag{4.25}$$

for all $\omega\in\Omega$ if $0<\varepsilon_1\le\varepsilon_2\le\rho_0$. And, in order to prove the existence of ED, we must just prove that $N^+_\varepsilon(\omega,\alpha_0)<N^-_\varepsilon(\omega,\alpha_0)$ for all $\omega\in\Omega$ if $\varepsilon\in(0,\rho_0)$: see Theorem 5.58 of [27], which ensures that in this case $M^\pm_\varepsilon(\omega,\alpha_0)=N^\pm_\varepsilon(\omega,\alpha_0)$ and hence that $M^+_\varepsilon(\omega,\alpha_0)<M^-_\varepsilon(\omega,\alpha_0)$ if $\varepsilon\in(0,\rho(\alpha_0))$. This will be done in point **7** below, after some preliminary work.

Let us consider the new auxiliary families

$$\mathbf{z}'=H^\lambda_\mu(\omega{\cdot}t)\,\mathbf{z}\,,\quad\text{where}\quad H^\lambda_\mu(\omega):=\begin{bmatrix}H_1(\omega) & H_3(\omega)+\mu I_n\\ H_2(\omega)-\alpha\Delta(\omega) & -H_1^T(\omega)\end{bmatrix}\mathbf{z}\,,\tag{4.26}$$

for $\mu\in\mathbb{C}$ and $\alpha\in\mathcal{I}$, which agree with $(4.3)^\alpha_\varepsilon$ if $\mu=\varepsilon\in\mathbb{R}$. If we take a real value of μ, say ε_0, in the interval $(0,\rho(\alpha_0))$, then $(4.26)^{\alpha_0}_{\varepsilon_0}$ has ED. Hence, according to Theorem 3.6(i), all the families $(4.26)_\mu^{\alpha_0}$ have ED if $\operatorname{Im}\mu > 0$, and the Weyl matrices $M_\mu^\pm(\omega,\alpha_0)$ determine Herglotz functions on $\mathbb{C}^+$ for each $\omega\in\Omega$ fixed, namely $\mu\mapsto M_\mu^\pm(\omega,\alpha_0)$, with $\pm\operatorname{Im}M_\mu^\pm(\omega,\alpha_0)\ge 0$. The same reason justifies the existence of the Weyl functions (with the same properties) $M_\mu^\pm(\omega,\alpha_m)$ for all $m\in\mathbb{N}$ if $\operatorname{Im}\mu>0$. Recall also that $(0,\rho_0)\subseteq(0,\rho(\alpha_m))$ for all $m\ge 1$. These facts allow will be used in the proof of the following assertion:

6. For all $\omega\in\Omega$ and all $\varepsilon\in(0,\rho_0)$ there exist the limits $\widetilde F_\varepsilon^\pm(\omega,\alpha_0):=\lim_{\beta\to0^+}M_{\varepsilon+i\beta}^\pm(\omega,\alpha_0)$, and they are real matrices. Moreover, the matrix-valued functions $t\mapsto\widetilde F_\varepsilon^\pm(\omega{\cdot}t,\alpha_0)$ are two globally defined solutions of the Riccati equation associated to $(4.26)_{\varepsilon_0}^\alpha$. In particular, $N_\varepsilon^+(\omega,\alpha_0)\le\widetilde F_\varepsilon^\pm(\omega,\alpha_0)\le N_\varepsilon^-(\omega,\alpha_0)$

Let us sketch this proof in the case of M^+: it adapts the arguments leading us to the proof of properties **3** and **4** in Theorem 4.5, where all the details are provided. We fix $\omega\in\Omega$ and $\varepsilon\in(0,\varepsilon_0)$, and represent

$$M_{\varepsilon+i\beta}^+(\omega,\alpha_m)=L_m+K_m\,(\varepsilon+i\beta)+\int_{\mathbb{R}}\left(\frac{1}{t-\alpha-i\beta}-\frac{t}{t^2+1}\right)dP_m(t)$$

for $m\ge1$ and

$$M_{\varepsilon+i\beta}^+(\omega,\alpha_0)=L_0+K_0\,(\varepsilon+i\beta)+\int_{\mathbb{R}}\left(\frac{1}{t-\varepsilon-i\beta}-\frac{t}{t^2+1}\right)dP_0(t)$$

Let us take $m\ge1$. Then

$$\frac12\left(P_m\{\varepsilon_1\}+P_m\{\varepsilon_2\}\right)+\int_{(\varepsilon_1,\varepsilon_2)}dP_m(t)=\frac1\pi\int_{\varepsilon_1}^{\varepsilon_2}\operatorname{Im}M_\varepsilon^+(\omega{\cdot}d,\alpha)\,d\varepsilon=0_n$$

whenever $0<\varepsilon_1<\varepsilon_2<\rho_0$. This ensures that $\int_{(0,\rho_0)}dP_m(t)=0_n$ if $m\ge1$, which together with the property $\lim_{m\to\infty}M_{\varepsilon+i\beta}^+(\omega,\alpha_m)=M_{\varepsilon+i\beta}^+(\omega,\alpha_0)$ uniformly on the compact sets of $\mathbb{C}^+$ allows us to deduce that (dP_m) converges to dP_0 in the weak* sense, and hence that $\int_{(0,\rho_0)}dP_0(t)=0_n$.

Let us take $\delta\in(0,\rho_0/2)$ and $\mathcal{C}_\delta:=\{\varepsilon+i\beta\in\mathbb{C}\mid\varepsilon\in[\delta,\rho_0-\delta]\text{ and }\beta\in(0,1]\}$. Note that there exist constants $c_1>0$ and $c_2>c_1$ such that

$$c_1\le\frac{|t-\varepsilon-i\beta|^2}{t^2+1}\le c_2\quad\text{for }\ t\notin(0,\rho_0)\ \text{ and }\ \varepsilon+i\beta\in\mathcal{C}_\delta\,,$$

which together wit the previous property leads to

$$0_n\le\operatorname{Im}M_{\varepsilon+i\beta}^+(\omega,\alpha_0)\le\beta\left(K_0+\frac{1}{c_1}\,\operatorname{Im}M_{\varepsilon+i}^+(\omega,\alpha_0)\right)$$

if $\varepsilon\in[\delta,\rho-\delta]$ and $\beta\in(0,1]$. In particular, $\lim_{\beta\to0^+}\operatorname{Im}M_{\varepsilon+i\beta}(\omega,\alpha_0)=0_n$ uniformly on the compact subsets of $(0,\rho_0)$. The Schwarz reflection principle allows us to ensure that there exists $\widetilde F_\varepsilon^+(\omega,\alpha_0):=\lim_{\beta\to0^+}M_{\varepsilon+i\beta}^+(\omega,\alpha_0)$, and it is a real matrix. The arguments used at the end of the proof of **4** and to obtain (4.18) complete the proof of **6** in the case of M^+. And the case of M^- is proved in the same way.

The information provided by **6** will allow us to adapt the proof of property **5** in Theorem 4.5 in order to conclude that

7. $N_\varepsilon^+(\omega,\alpha_0)<N_\varepsilon^-(\omega,\alpha_0)$ for all $\omega\in\Omega$ if $\varepsilon\in(0,\rho_0)$,

which, as said before, shows the existence of ED for $\varepsilon \in (0, \rho_0)$.

We proceed by contradiction, assuming the existence of $\widetilde{\omega} \in \Omega$, $\widetilde{\varepsilon} \in (0, \rho_0)$, and $\mathbf{v} \in \mathbb{R}^n - \{\mathbf{0}\}$ such that $\mathbf{v}^T \big(N^-_{\widetilde{\varepsilon}}(\widetilde{\omega}, \alpha_0) - N^+_{\widetilde{\varepsilon}}(\widetilde{\omega}, \alpha_0)\big)\,\mathbf{v} = 0$. This fact and (4.25) ensure that

$$\mathbf{v}^T \big(N^-_{\varepsilon}(\widetilde{\omega}, \alpha_0) - N^+_{\varepsilon}(\widetilde{\omega}, \alpha_0)\big)\,\mathbf{v} = 0 \quad \text{if } \varepsilon \in (\widetilde{\varepsilon}, \rho_0)\,. \tag{4.27}$$

Now we define $h\colon \mathbb{C}^+ \to \mathbb{C}$ by

$$h(\varepsilon + i\beta) := \mathbf{v}^T \big(M^-_{\varepsilon+i\beta}(\omega, \alpha_0) - M^+_{\varepsilon+i\beta}(\omega, \alpha_0)\big)\,,$$

which is holomorphic. In addition, since property **6** combined with Theorem 5.48(iii) of [27] ensures that

$$N^+_{\varepsilon}(\widetilde{\omega}, \alpha_0) \le \widetilde{F}^{\pm}_{\varepsilon}(\widetilde{\omega}, \alpha_0) \le N^-_{\varepsilon}(\widetilde{\omega}, \alpha_0) \quad \text{if } \varepsilon \in (\widetilde{\varepsilon}, \rho_0]\,,$$

we deduce from (4.27) that

$$\lim_{\beta\to 0^+} h(\varepsilon + i\beta) = \mathbf{v}^T \big(\widetilde{F}^-_{\varepsilon}(\widetilde{\omega}, \alpha_0) - \widetilde{F}^+_{\varepsilon}(\widetilde{\omega}, \alpha_0)\big)\,\mathbf{v} = 0 \quad \text{for } \varepsilon \in (\widetilde{\varepsilon}, \rho_0)\,.$$

Hence, there exists a holomorphic extension $\widetilde{h}$ of h to the set $(\mathbb{C} - \mathbb{R}) \cup (\widetilde{\varepsilon}, \rho_0)$ such that $\widetilde{h}(\widetilde{\omega}, \varepsilon) = 0$ for $\varepsilon \in \mathbb{R} - [\widetilde{\varepsilon}, \rho_0]$. The principle of isolated zeroes ensures then that

$$\widetilde{h}(\varepsilon + i\beta) = \mathbf{v}^T \big(M^-_{\varepsilon+i\beta}(\widetilde{\omega}, \alpha_0) - M^+_{\varepsilon+i\beta}(\widetilde{\omega}, \alpha_0)\big)\,\mathbf{v} = 0 \quad \text{if } \varepsilon \in \mathbb{R} \text{ and } \beta > 0\,.$$

Taking now limits at a point $\varepsilon \in (0, \rho(\alpha_0))$ yields $\mathbf{v}^T \big(M^-_{\varepsilon}(\widetilde{\omega}, \alpha_0) - M^+_{\varepsilon}(\widetilde{\omega}, \alpha_0)\big)\,\mathbf{v} = 0$. But this is impossible, since, as seen before, $M^-_{\varepsilon}(\widetilde{\omega}, \alpha_0) - M^+_{\varepsilon}(\widetilde{\omega}, \alpha_0) > 0$. This is the sought-for contradiction: the proofs of **7** and (ii) are complete.

(iii) The first assertion in (iii) has been checked at the beginning of the first step in the proof of Theorem 4.5. In order to check that ρ is injective at the (perhaps nonexistent) interval at which it takes real values, we take $\widetilde{\alpha} \in \mathcal{I}$ such that $\widetilde{\rho} := \rho(\widetilde{\alpha}) < \infty$. Let us consider the families $(4.3)^{\alpha}_{\widetilde{\rho}}$ for α varying in $\mathbb{R}$. Since $(4.3)^{\widetilde{\alpha}}_{\widetilde{\rho}}$ is UWD, Theorem 5.61 of [27] ensures that $(4.3)^{\widetilde{\alpha}}_{\widetilde{\rho}}$ has ED and satisfies the NC for $\alpha < \widetilde{\alpha}$. And this ensures that $\rho(\alpha) > \widetilde{\rho}$ if $\alpha < \widetilde{\alpha}$, which proves the injectivity of the map.

(iv) Assume for contradiction that $\alpha^* < \infty$ and that the family $(4.2)^{\alpha^*}$ has ED. Then there exist $\varepsilon_0 > 0$ and $\alpha_0 < \alpha^*$ such that the families $(4.3)^{\alpha}_{\varepsilon}$ have ED if $\varepsilon \in [0, \varepsilon_0]$ and $\alpha \in [\alpha_0, \alpha^*]$: see Remark 2.4. Consequently, since the map ρ is decreasing, the families $(4.3)^{\alpha}_{\varepsilon}$ are UWD for $\varepsilon \in (0, \varepsilon_0]$ and $\alpha \in [\alpha_0, \alpha^*]$, and there exist $M^+_{\varepsilon}(\omega, \alpha) = N^+_{\varepsilon}(\omega, \alpha)$: see Remark 5.22 and Theorem 5.66 of [27]. Proposition 5.51 of [27] ensures that the two-parametric family $M^+_{\varepsilon}(\omega, \alpha)$ increases with α and decreases with ε. Let us take a sequence $(\varepsilon_m) \downarrow 0$ and a point $\alpha_1 \in (\alpha_0, \alpha^*)$. Then $M^+(\omega, \alpha_1) = \lim_{m\to\infty} M^+_{\varepsilon_m}(\omega, \alpha_1)$, so that

$$M^+(\omega, \alpha_1) \le M^+_{\varepsilon_m}(\omega, \alpha^1) \le M^+_{\varepsilon_m}(\omega, \alpha^*)$$

for all $m \ge 1$. In particular, the sequence of matrices $(M_{\varepsilon_m}(\omega, \alpha^*))$, which decreases, is bounded from below. Therefore, for any $\omega \in \Omega$, there exists a suitable convergent subsequence (use the polarization formulas). The continuous variation of the Lagrange planes associated to the ED ensures that this limit is necessarily $M^+(\omega, \alpha^*)$. Consequently, the family $(4.2)^{\alpha^*}$ satisfies NC, which ensures that $\alpha^* \in \mathcal{I}$. This is the sought-for contradiction. The proof is complete. □

Note that, under Hypotheses (4.1), the set $\mathcal{I}$ defined by (4.7) can either be upper bounded or agree with the whole real line. And, if it is bounded, then the value of $\rho_0 := \lim_{\alpha\to(\sup\mathcal{I})^-}\rho(\alpha)$ can be 0, a positive real value, or ∞. We check these assertions by means of simple autonomous examples:

- In the case of $\mathbf{z}' = \left[\begin{smallmatrix} -1 & 0 \\ -\lambda & 1\end{smallmatrix}\right]\mathbf{z}$, $\mathcal{I}=\mathbb{R}$ and $\rho_0=0$. More precisely, $M^+(\lambda) = \lambda/2$ (and $l^-(\lambda)=\{\left[\begin{smallmatrix} 0 \\ x\end{smallmatrix}\right] \mid x\in\mathbb{R}\}$, so that M^- does not exist) for all $\lambda\in\mathbb{R}$; and with $\rho(\lambda)=\infty$ for $\lambda\le 0$ and $\rho(\lambda)=1/\lambda$ for $\lambda>0$.
- In the case $\mathbf{z}' = \left[\begin{smallmatrix} -1 & 1 \\ -\lambda & 1\end{smallmatrix}\right]\mathbf{z}$, $\mathcal{I}=(-\infty,1)$ and $\rho_0=0$. More precisely, $\rho(\lambda)=\infty$ for $\lambda\le 0$ and $\rho(\lambda)=-1+1/\lambda$ for $\lambda\in(0,1)$.
- In the case $\mathbf{z}' = \left[\begin{smallmatrix} 0 & 1 \\ 1-\lambda & 0\end{smallmatrix}\right]\mathbf{z}$, $\mathcal{I}=(-\infty,1)$ and $\rho_0=\infty$, since $\rho(\lambda)=\infty$ for $\lambda<1$.
- Finally, combining the first and third examples, we obtain the 4-dimensional system
$$\mathbf{z}' = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 0 & 0 \end{bmatrix}\mathbf{z}\,,$$
for which $\mathcal{I}=(-\infty,1)$ and $\rho_0=\lim_{\alpha\to 1^-}1/\alpha = 1$.

However, if $\sup\mathcal{I}=\infty$, then $\lim_{\alpha\to(\sup\mathcal{I})^-}\rho(\alpha)=0$ (as it happens in the first example). This assertion follows from Theorem 4.3(ii) and Theorem 5.61 of [27] applied to the families $(4.3)_\alpha^{\varepsilon_0}$ for a fixed $\varepsilon_0>0$ and α varying in $\mathbb{R}$. The same results, combined with the robustness of the properties of ED and global existence of M^+ (see Remark 2.4) ensure that, if $\mathcal{I}$ is bounded and $\rho_0>0$, then the families $(4.3)_{\rho_0}^{\varepsilon}$ corresponding to $\varepsilon\in(0,\rho_0)$ are UWD but they do not have ED.

Remark 4.7. Theorem 4.4 can be easily extended to the case of *distallity* of the base flow (Ω,σ); that is, if the distance between $\omega_1\cdot t$ and $\omega_2\cdot t$ is strictly positive for all $t\in\mathbb{R}$ whenever $\omega_1\neq\omega_2$. In this case, Ω decomposes in the disjoint union of a family of (distal) minimal sets (see Ellis [8]). Therefore, we can apply Theorem 4.4 over each minimal component. Since the ED of a linear family is equivalent to the ED of each one of its systems (see Remark 2.2.2), and obviously the same happens with the existence of M^+, we conclude that the conclusion of the theorem also hold over the whole of Ω.

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Departamento de Matemática Aplicada, EII, Universidad de Valladolid, Paseo del Cauce 59, 47011 Valladolid, Spain.
E-mail address, Carmen Núñez: carnun@wmatem.eis.uva.es
E-mail address, Rafael Obaya: rafoba@wmatem.eis.uva.es