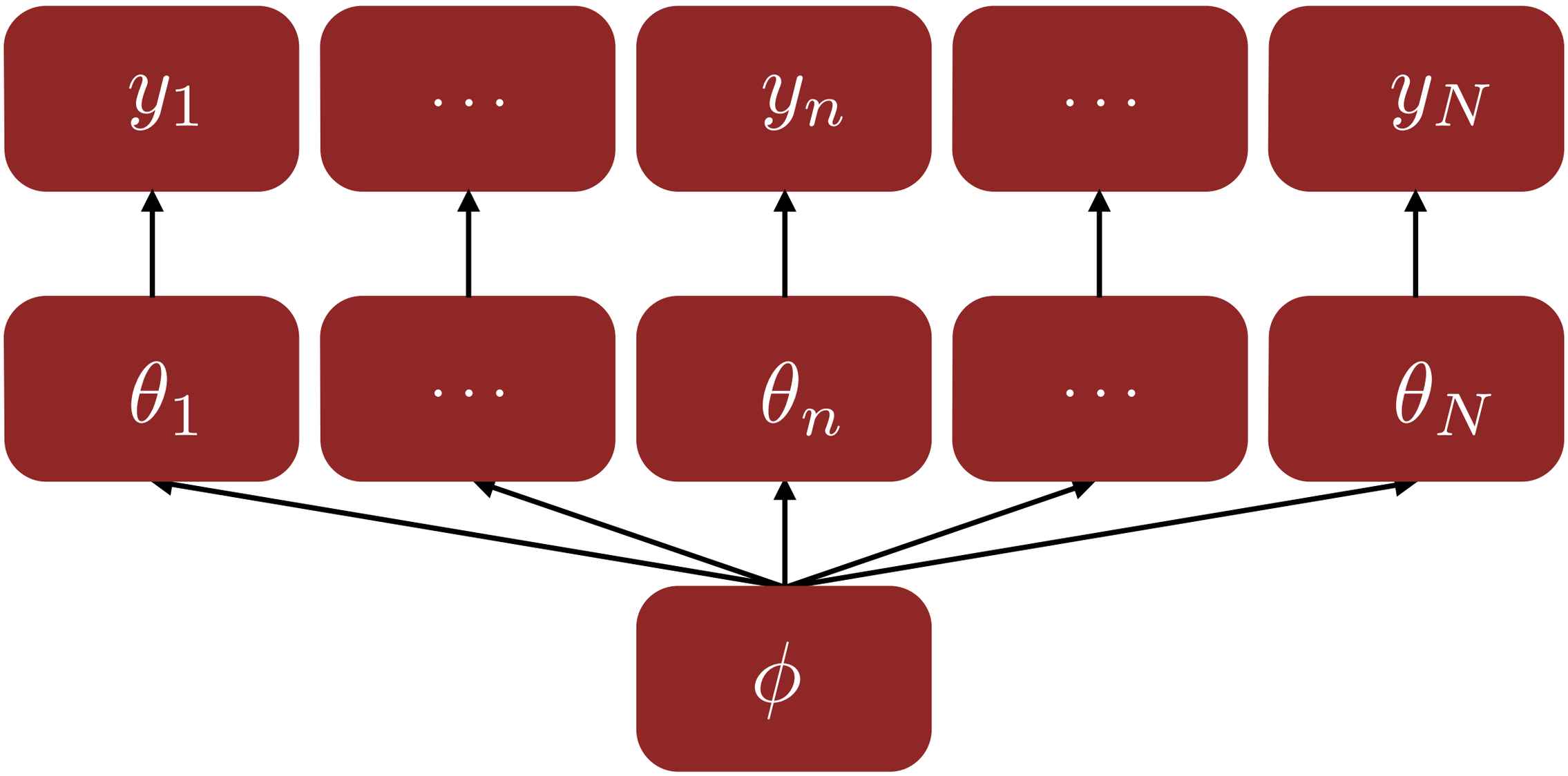
Hierarchical Models



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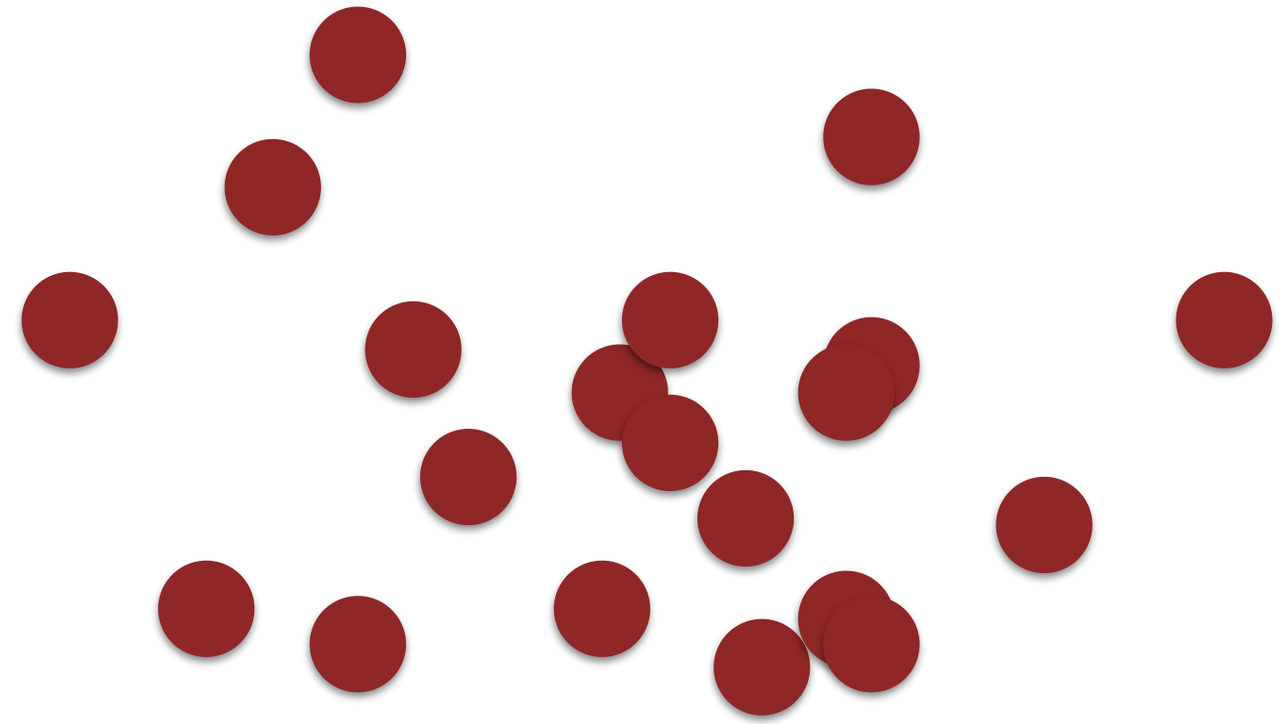

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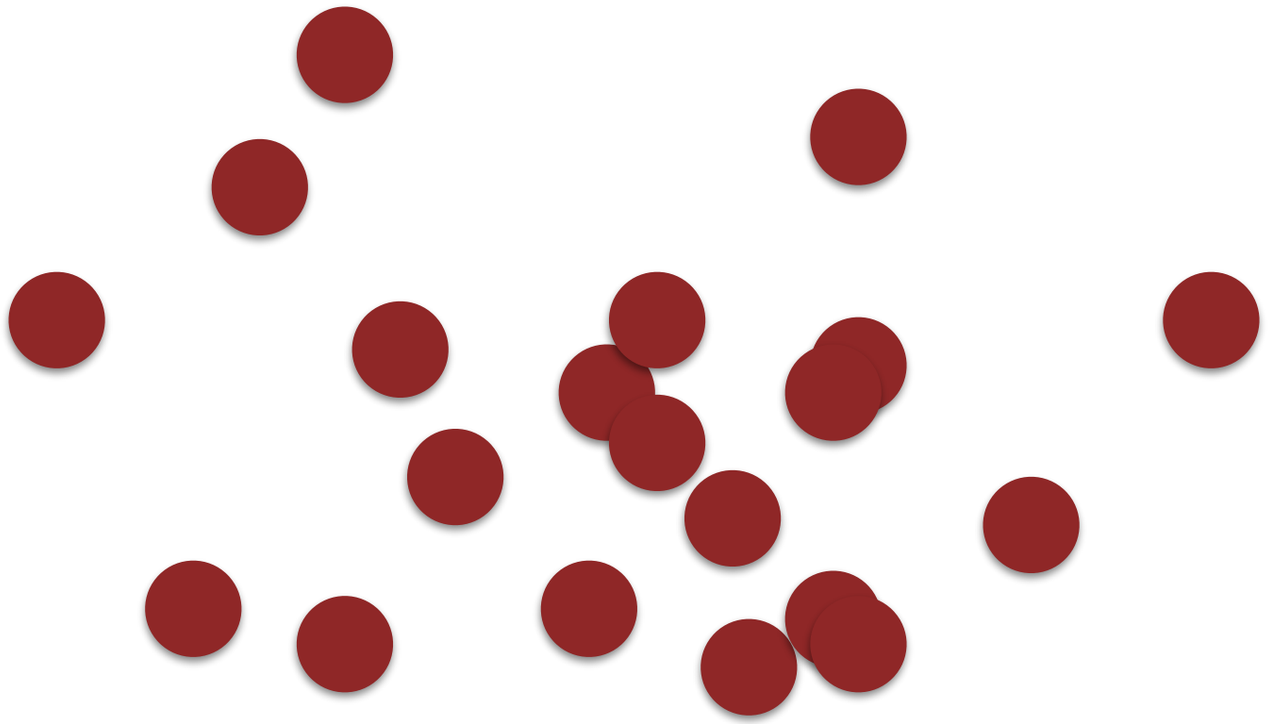
If we assume that every individual is equivalent then we can pool the data, but only at the expense of bias.



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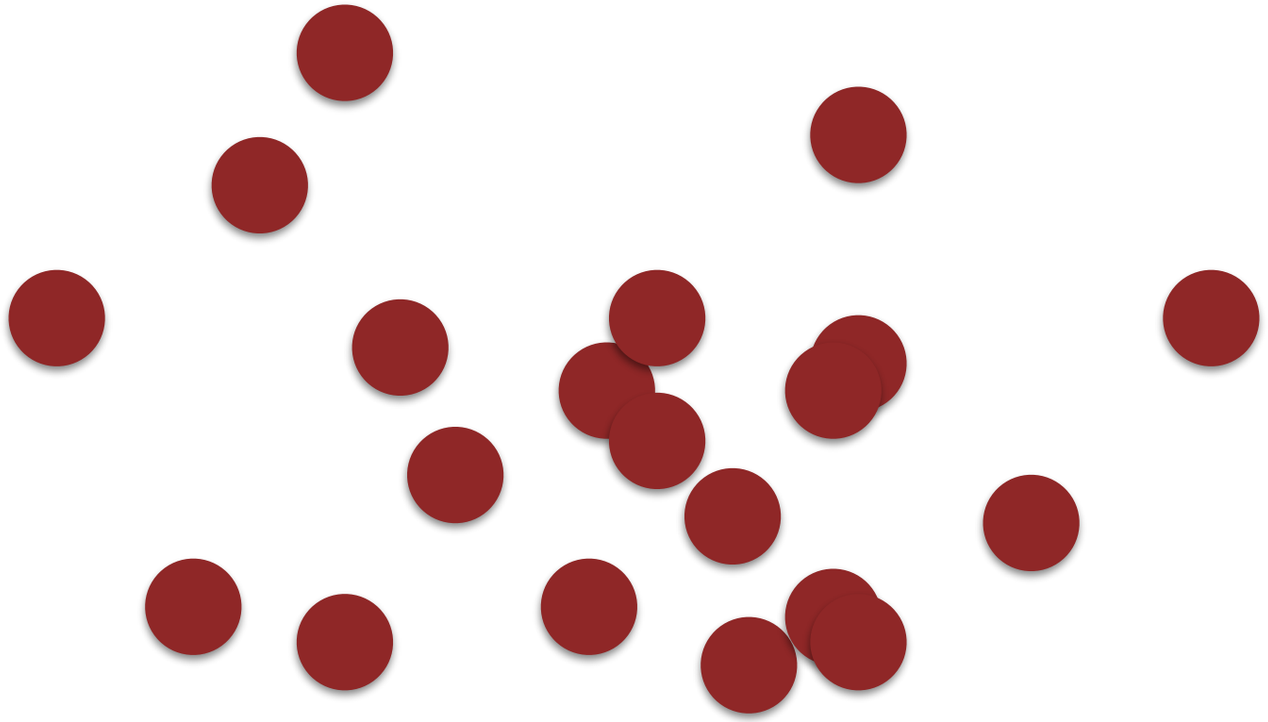
$$y_n \sim \mathcal{N}\left(\mathbf{X}^T \boldsymbol{\beta} + \alpha, \sigma\right)$$

Modeling every individual separately avoids any bias, but then the data becomes very sparse and inferences weak.



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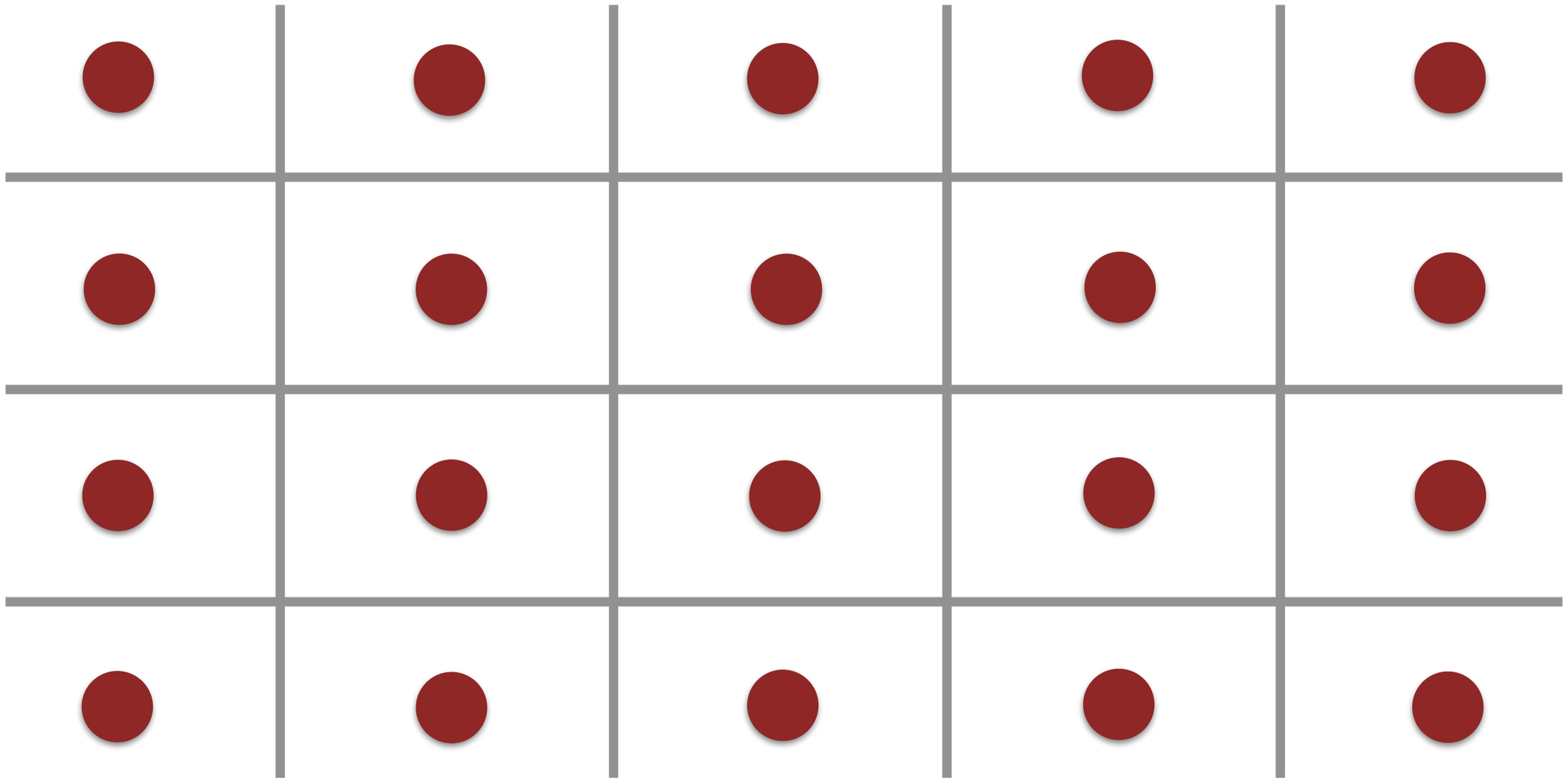

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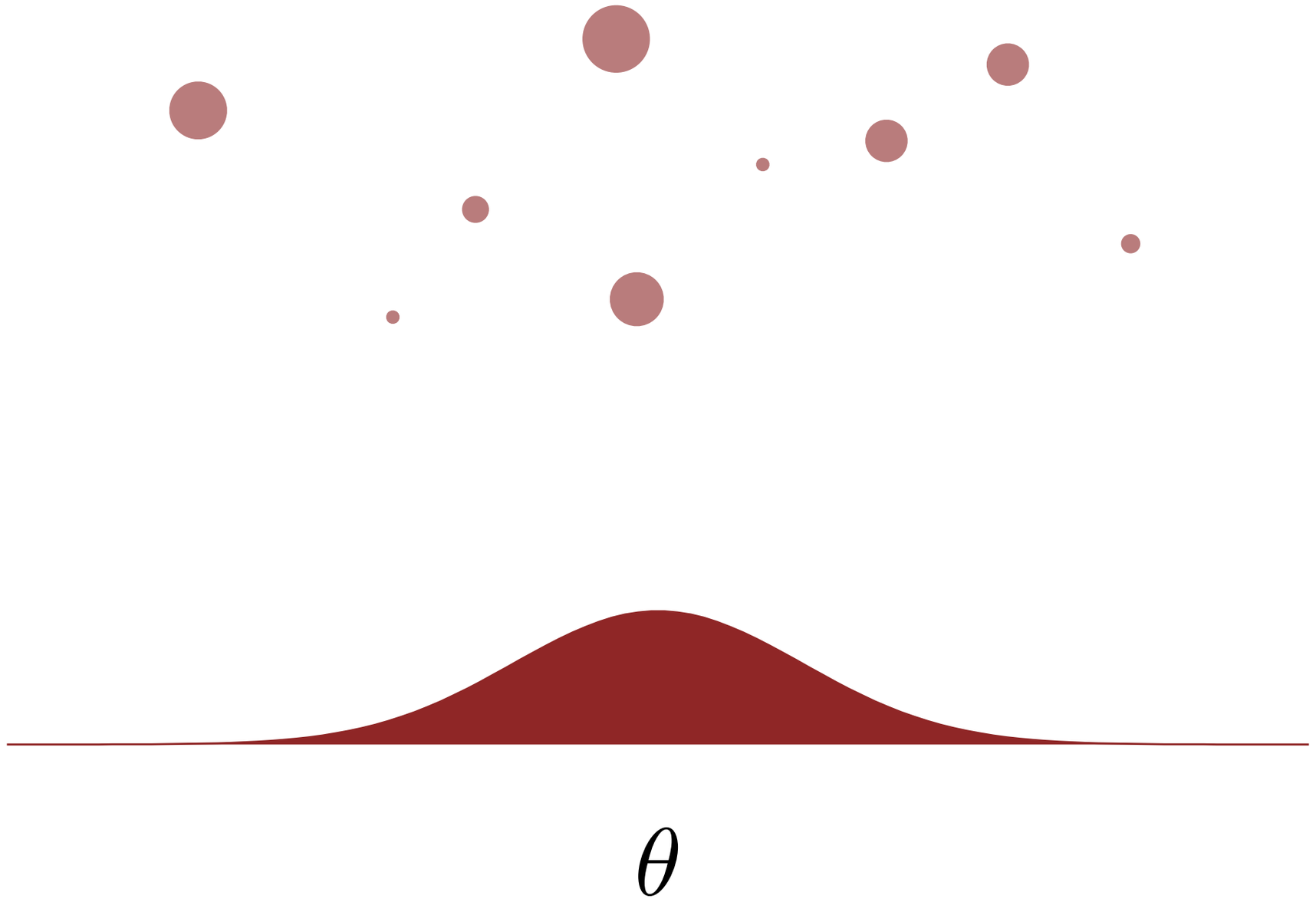

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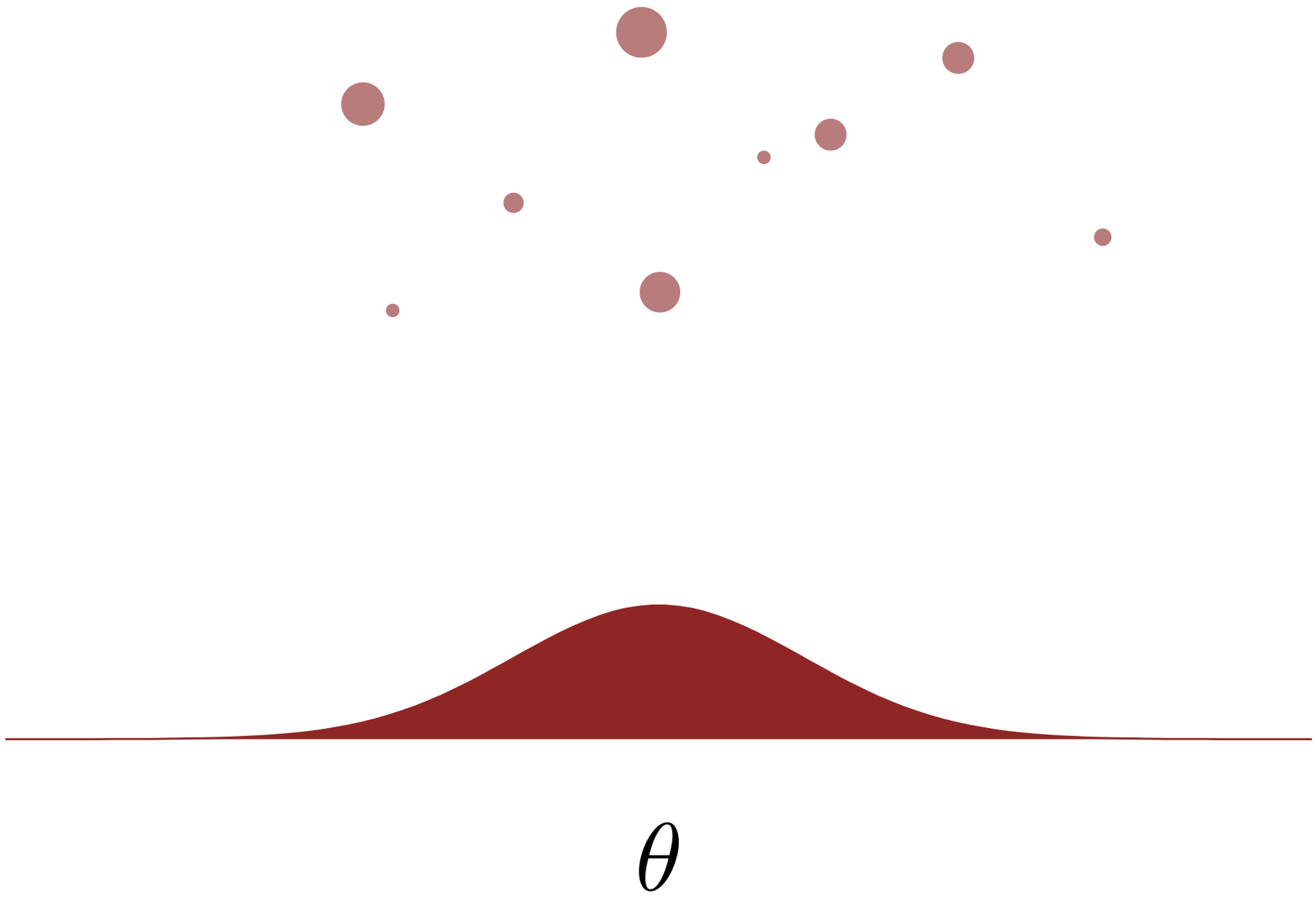
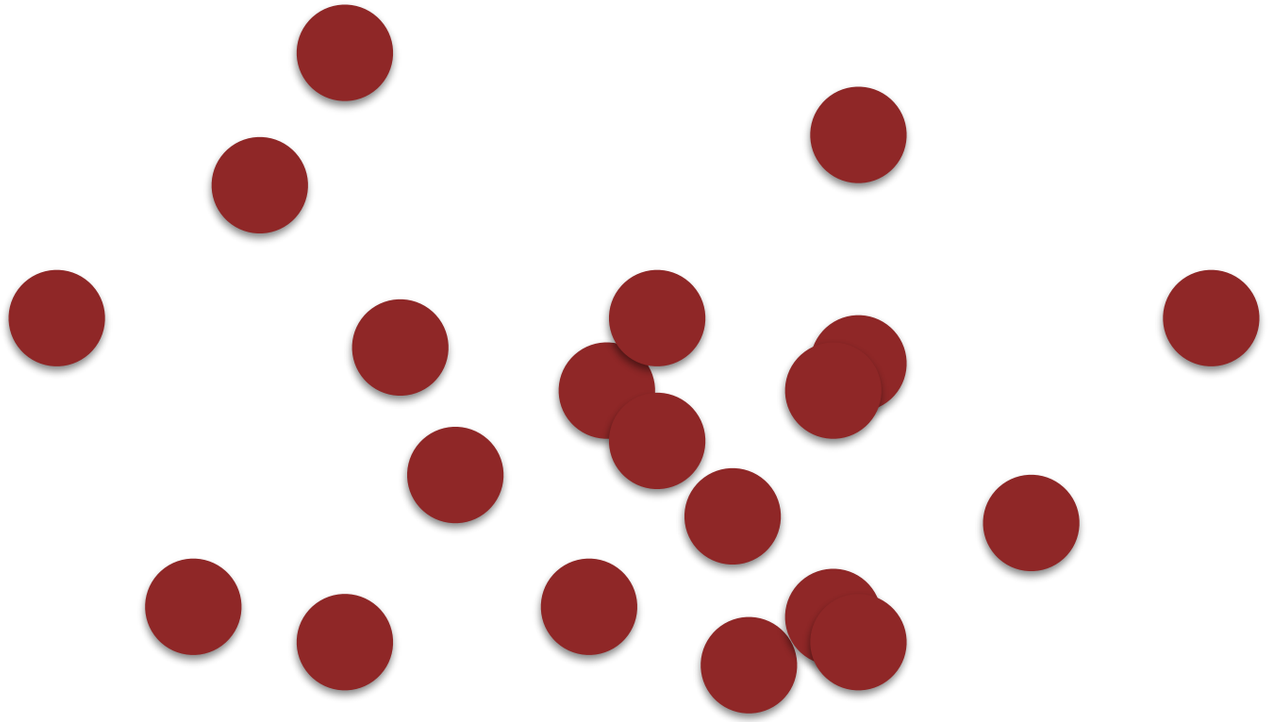


In order to formalize this approach we need to consider *exchangeabilitiy*.



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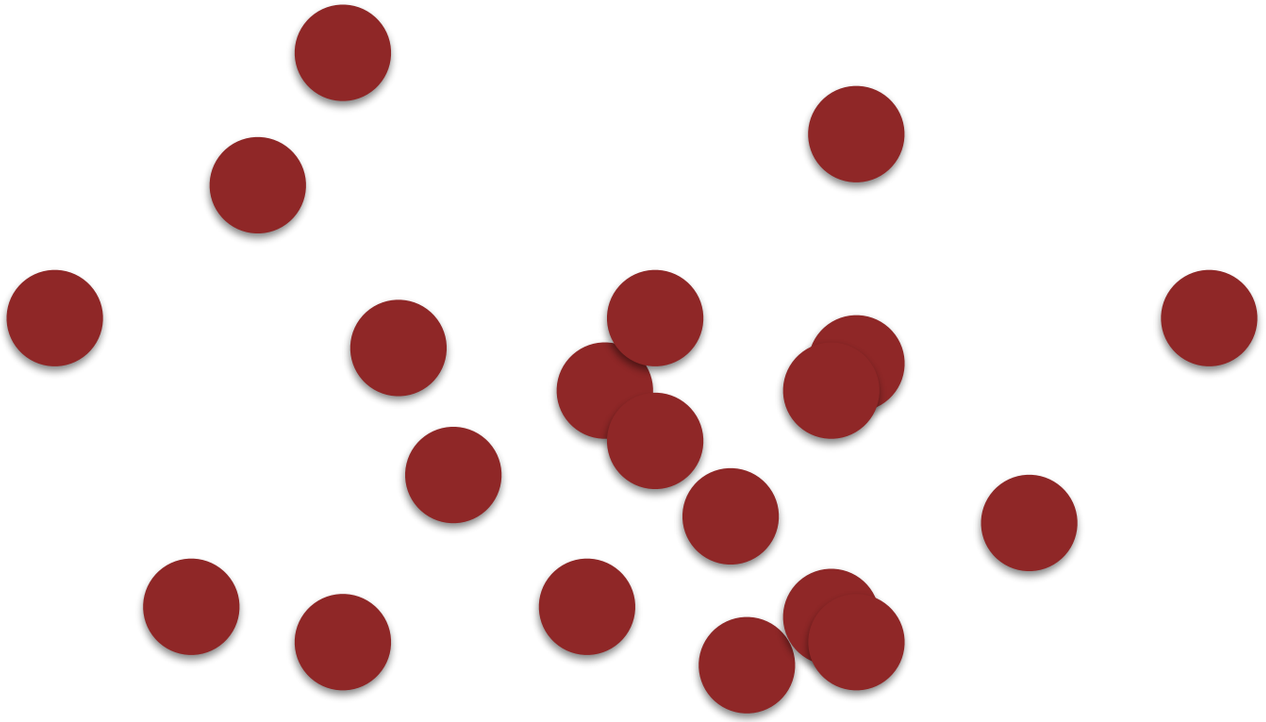

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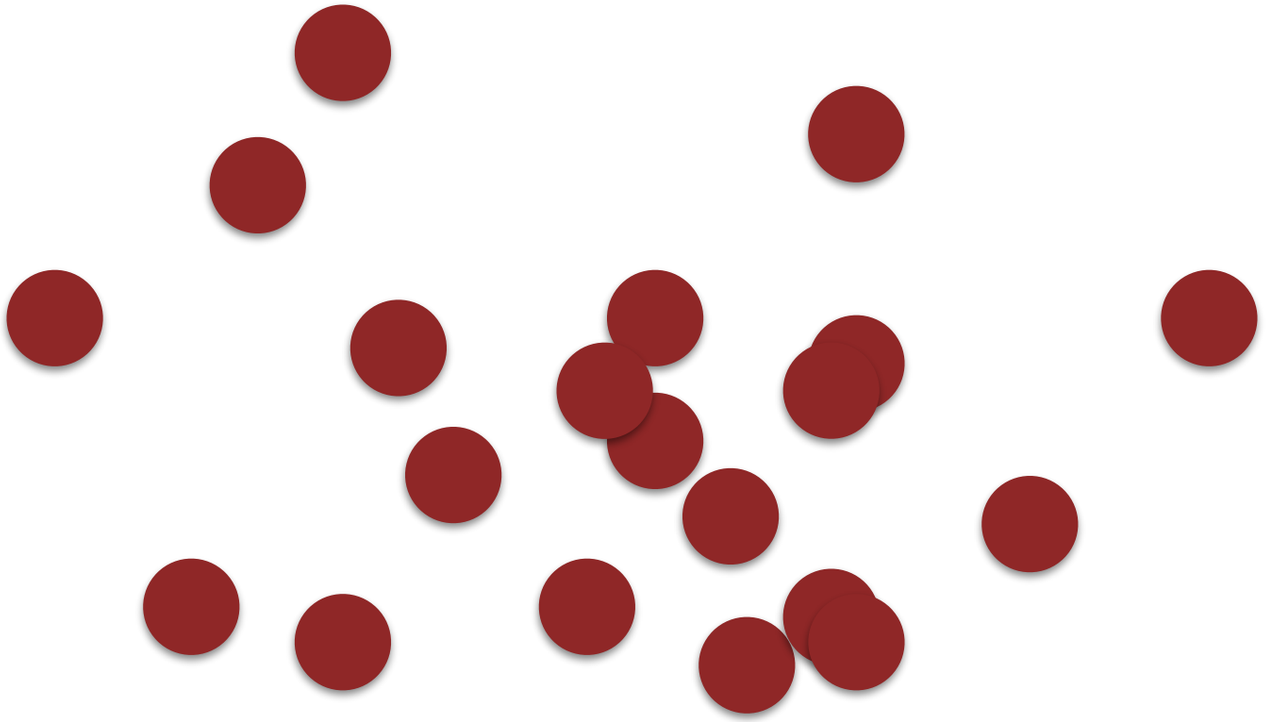
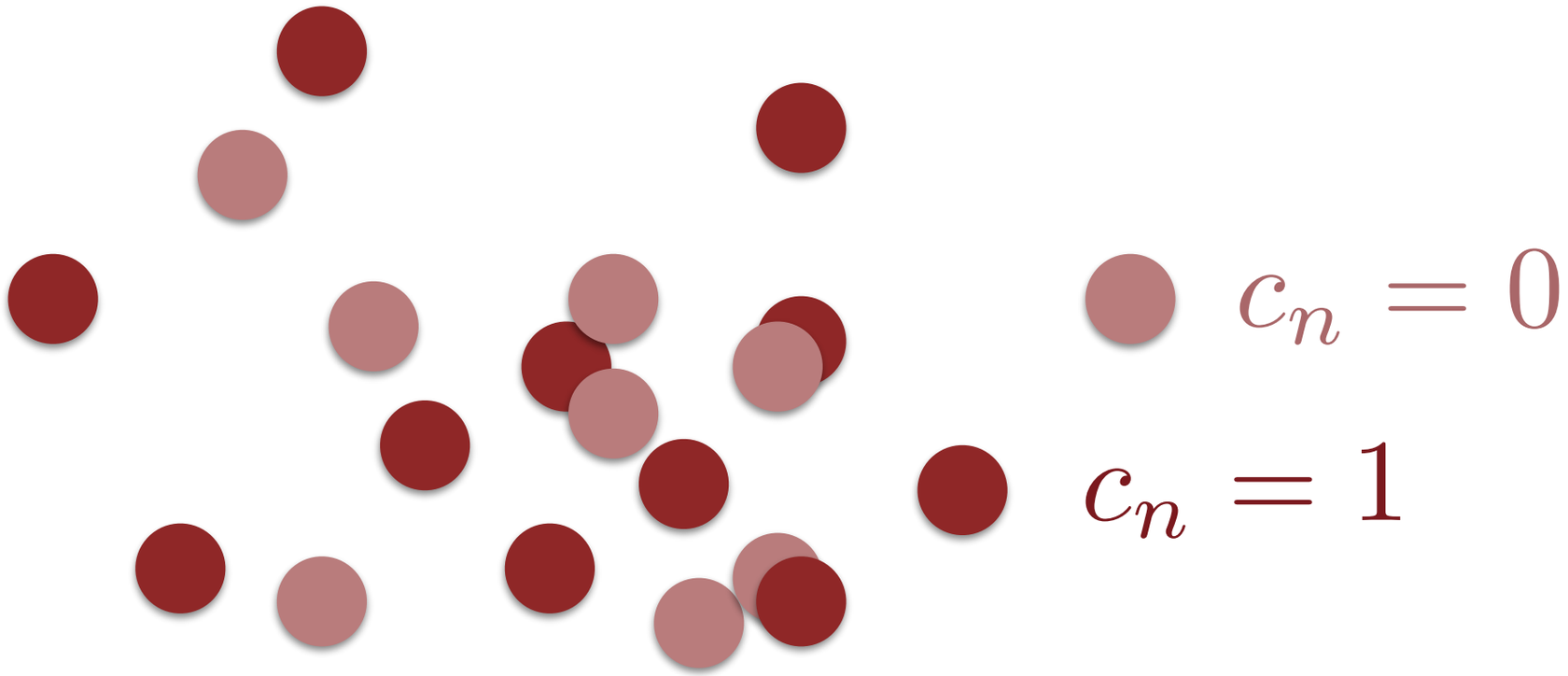


Given labels that discriminate between individuals,
the individuals are no longer exchangeable.



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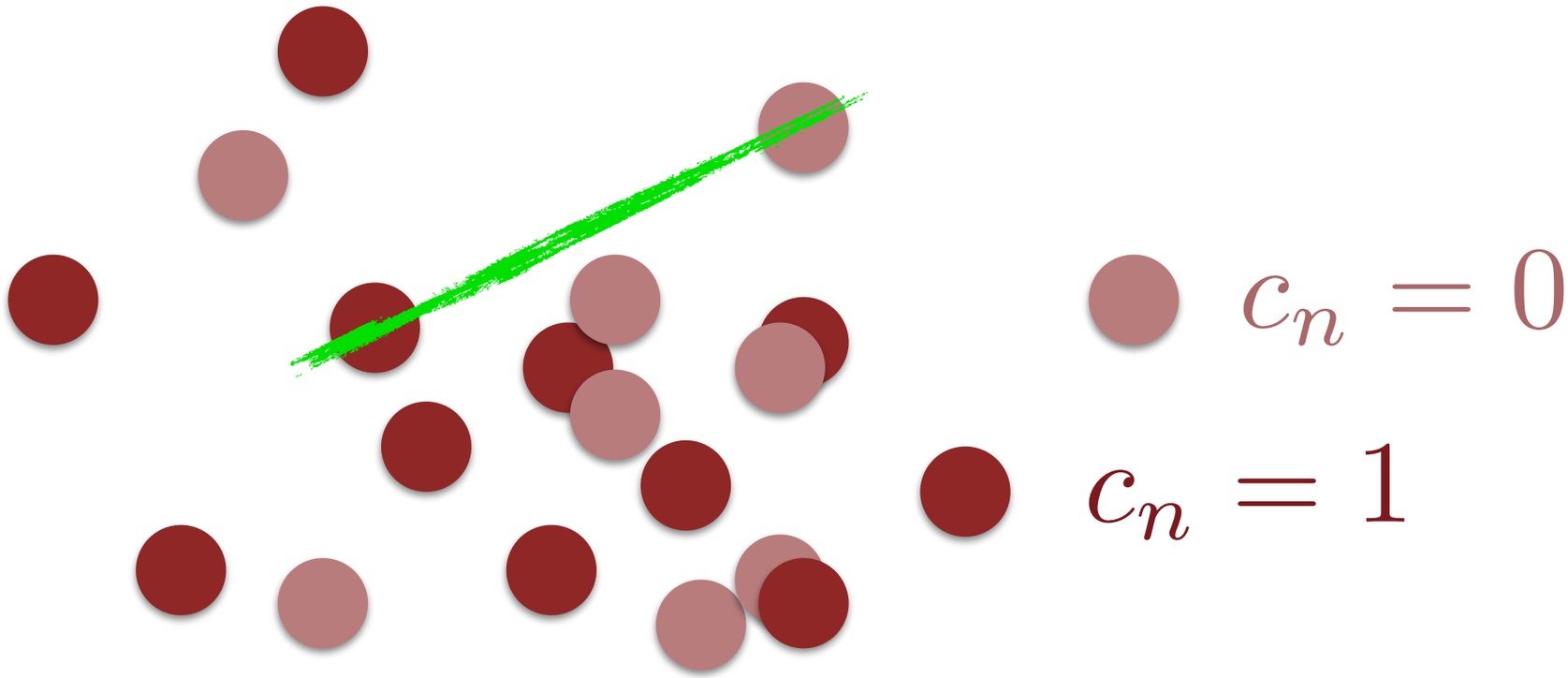
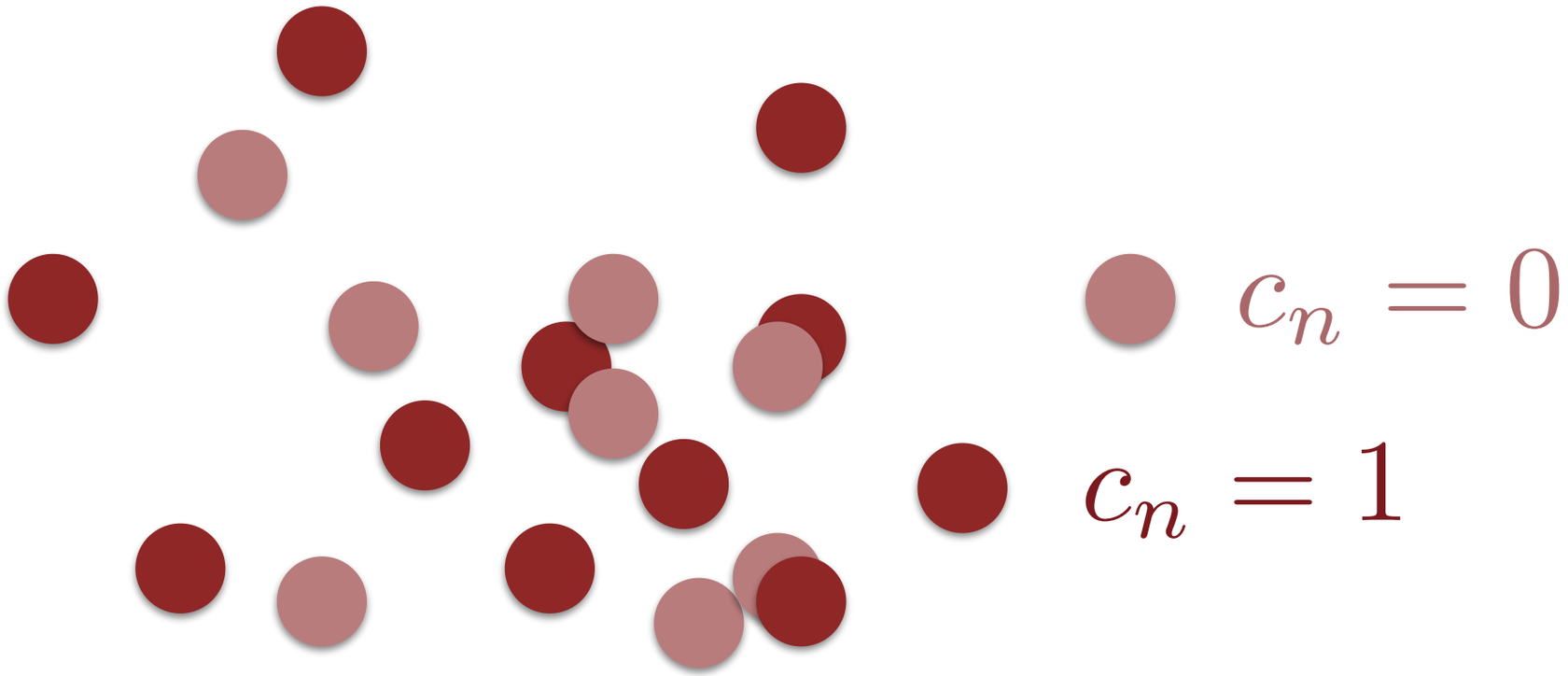


The partitions of the population, however, define exchangeable groups within a metapopulation.



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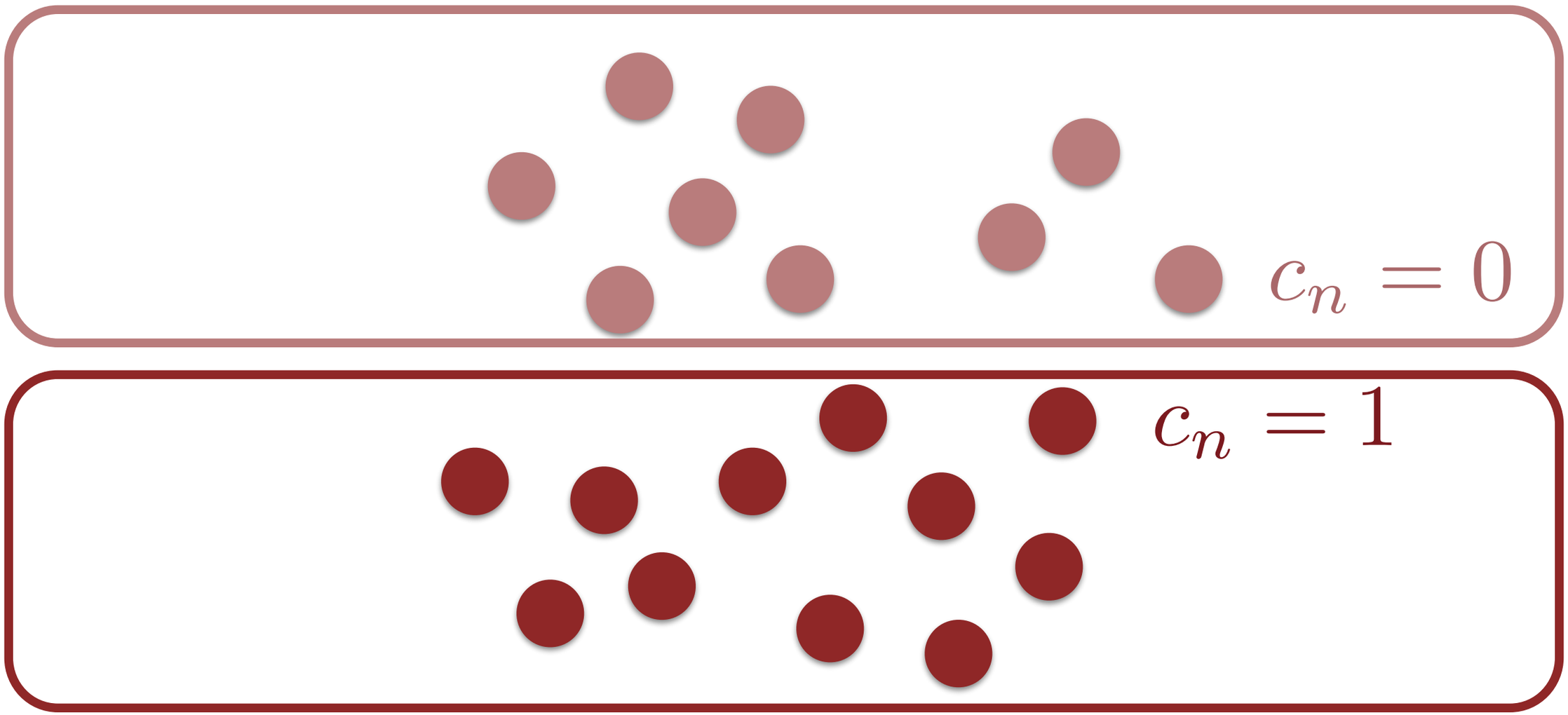

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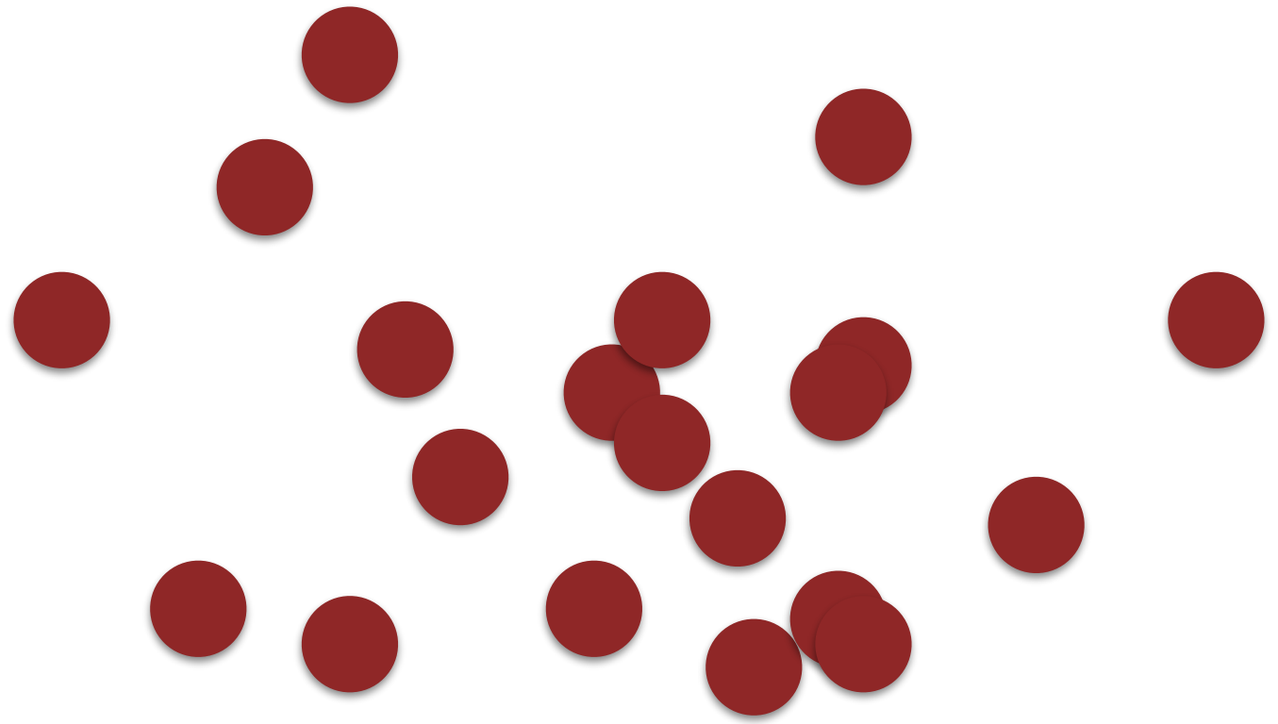

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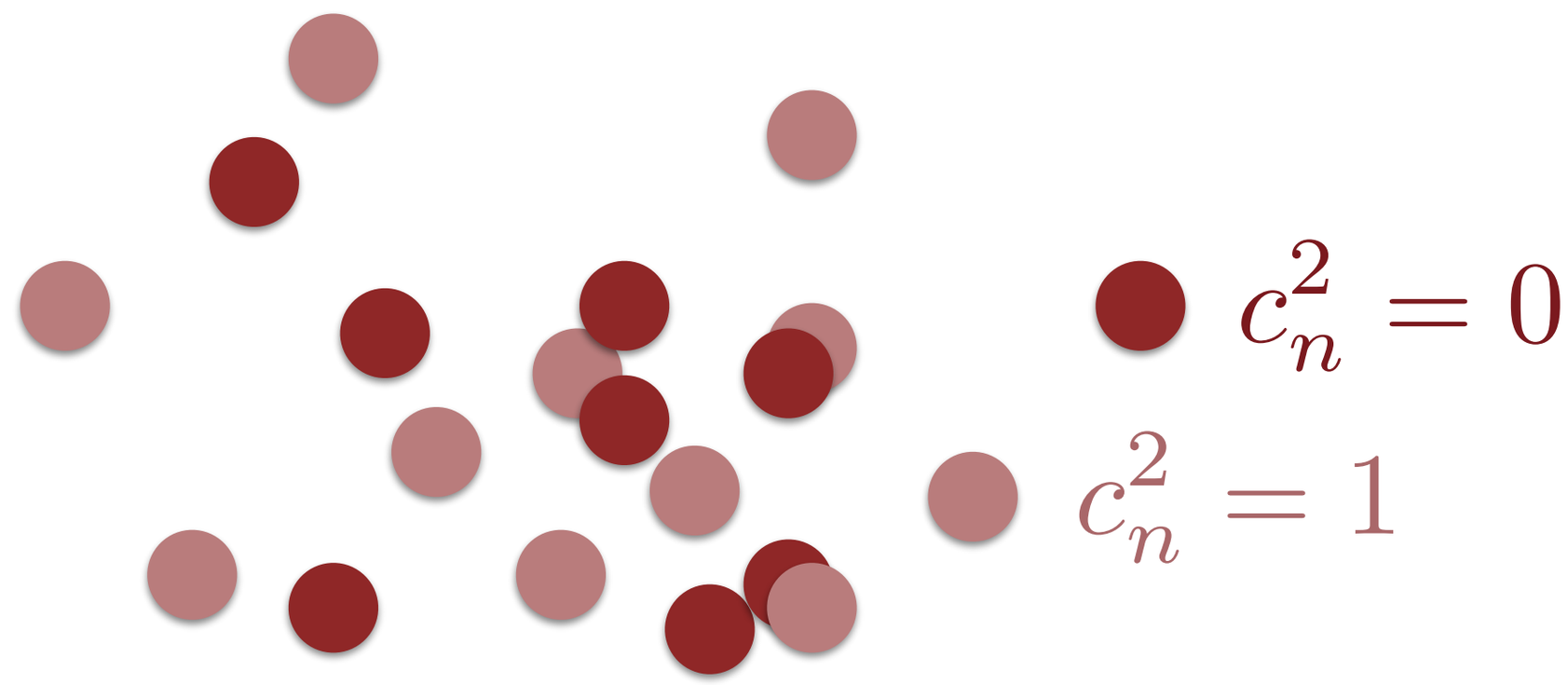

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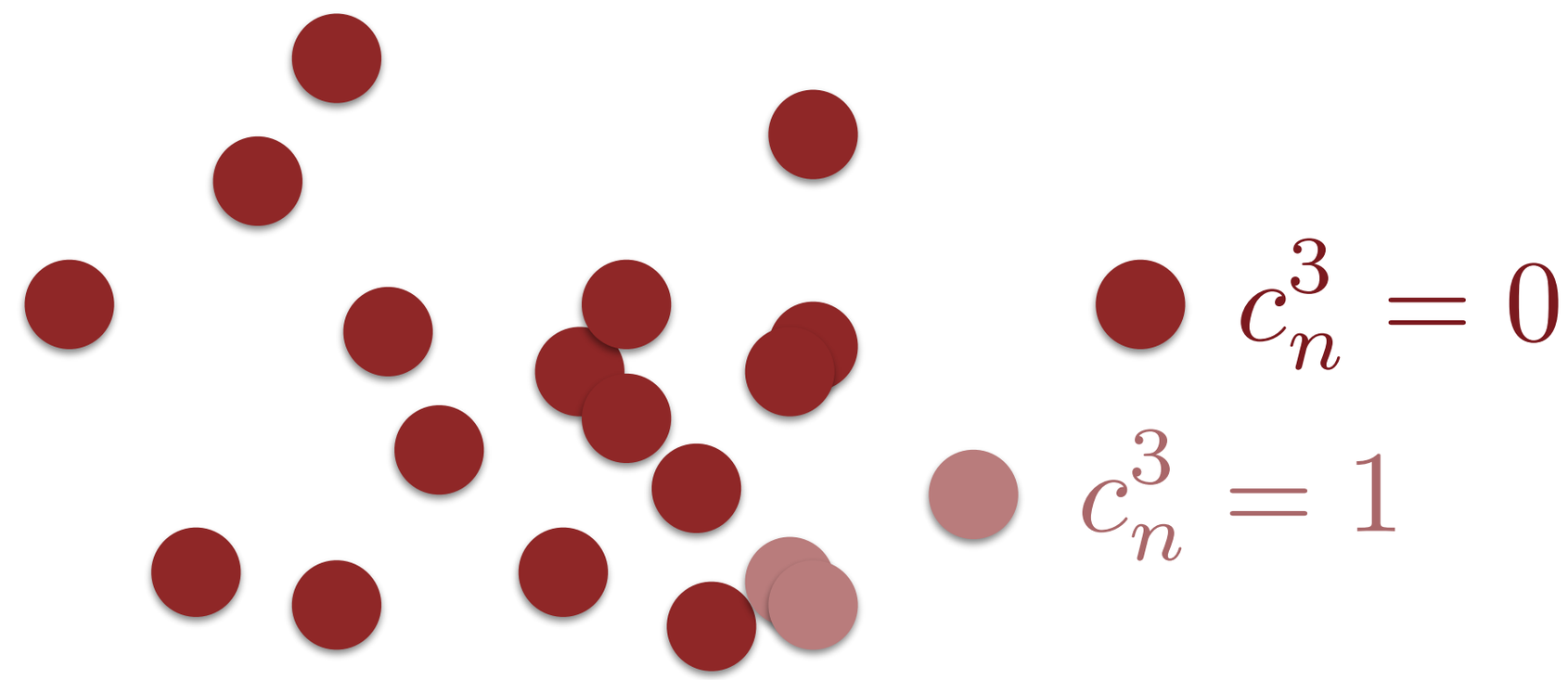

As the population grows, the only prior distribution that respects exchangeability is a *hierarchical* distribution.

$$\pi(\boldsymbol{\theta}) = \int \mathrm{d}\phi \prod_{n=1}^{N} \pi(\theta_n|\phi)\, \pi(\phi)$$

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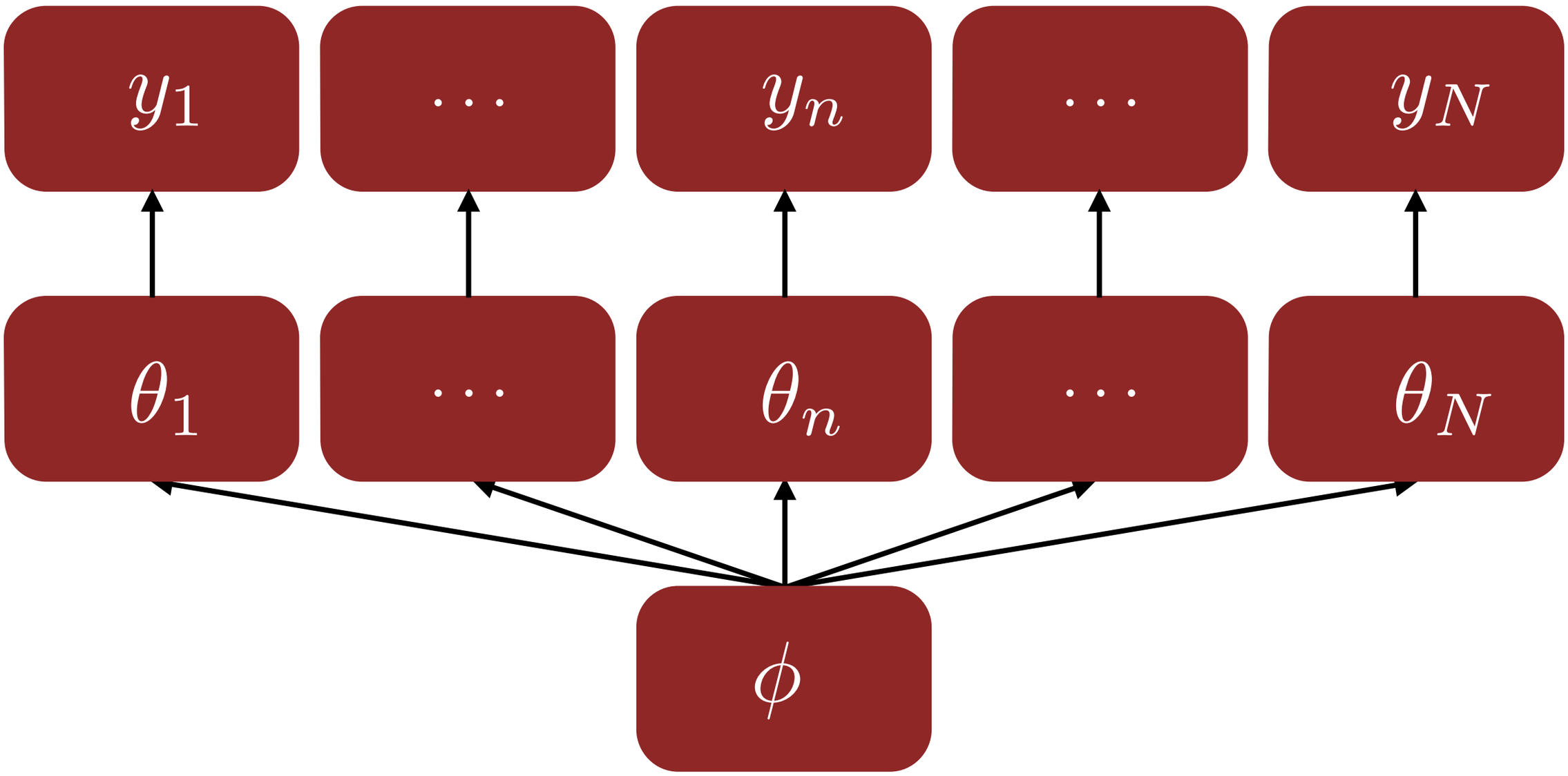

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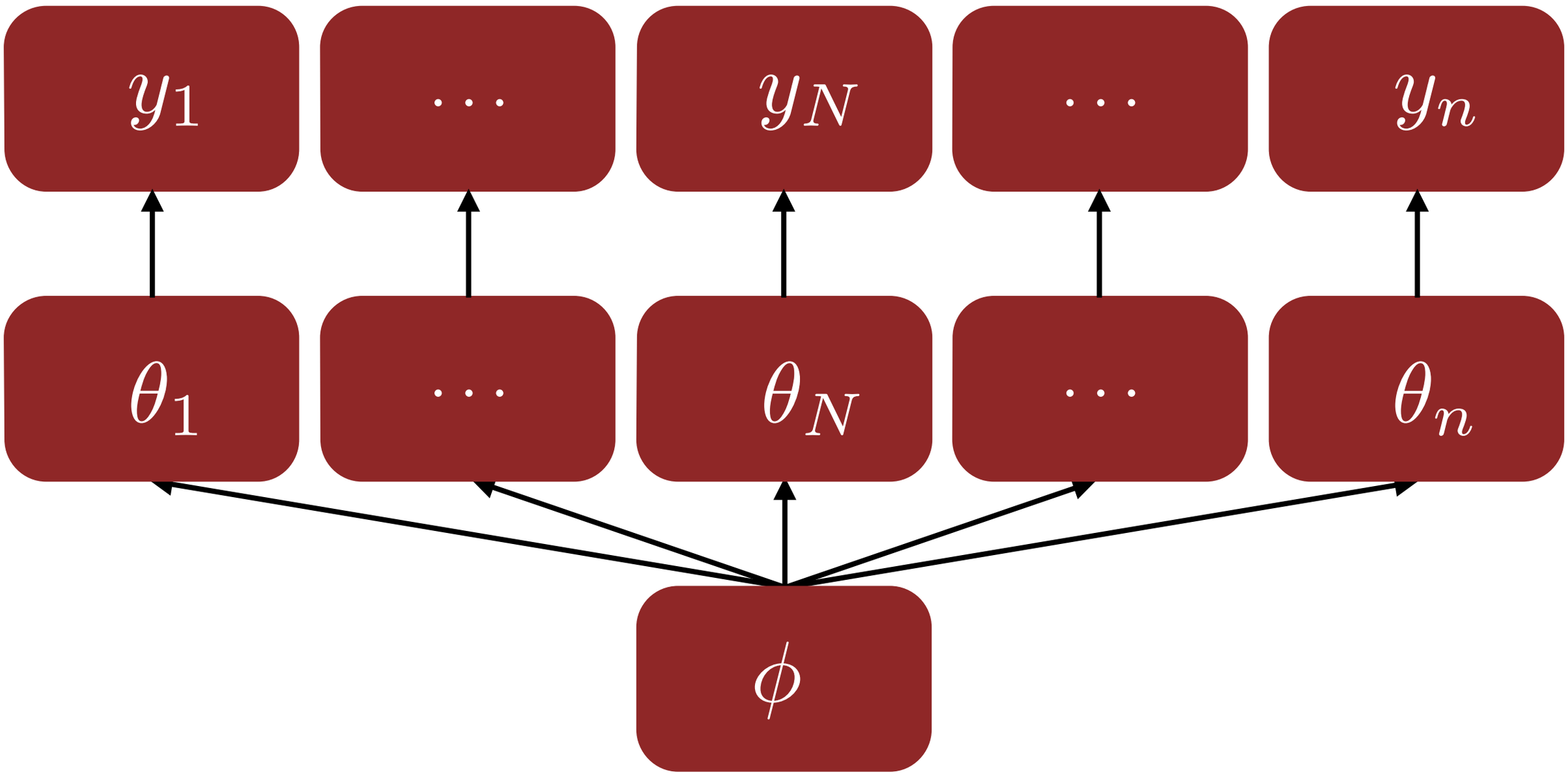
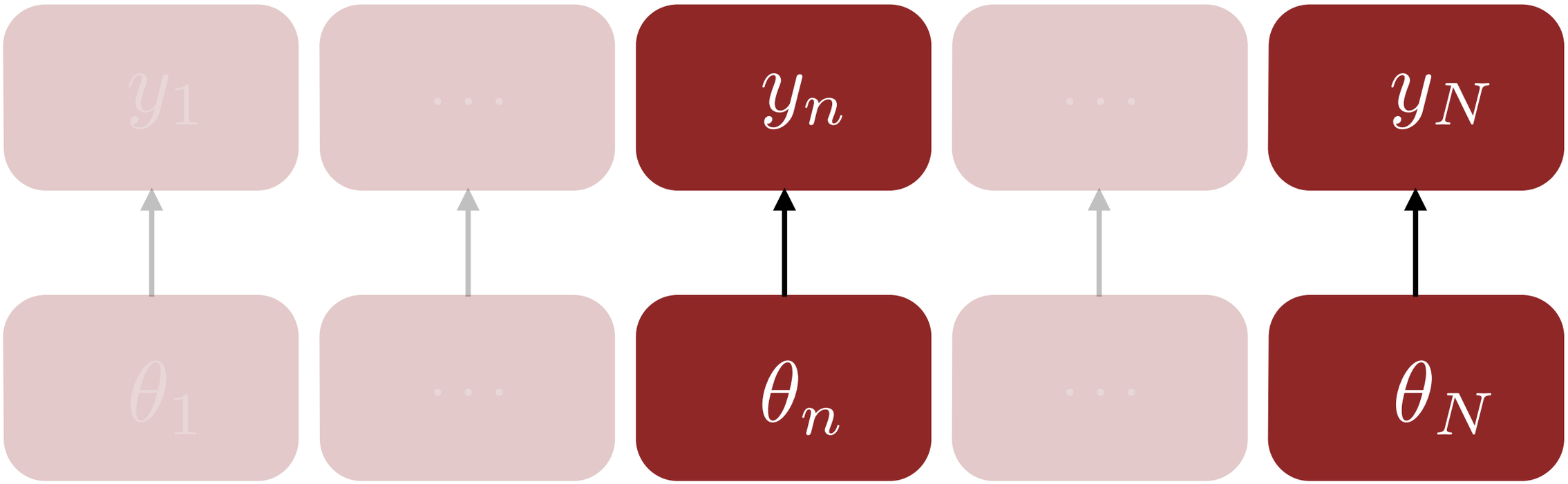


The hyperparameters couple all of the groups together, partially pooling the data and balancing bias and variance.



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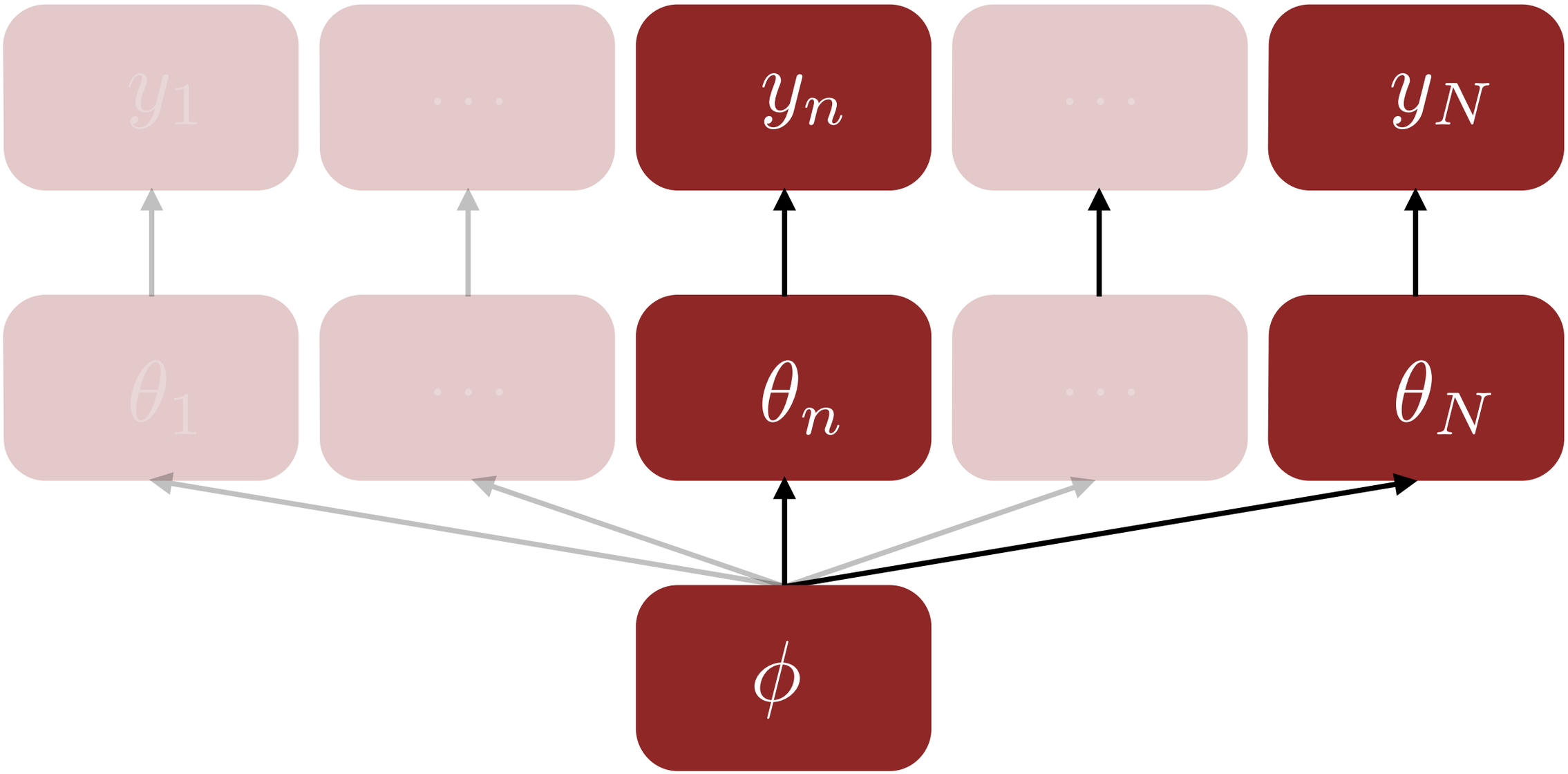

The most common population model is a Gaussian, where we model the population mean and variance.

$$\prod_{n=1}^{N} \pi(y_n|\theta_n)\,\pi(\theta_n|\phi)\,\pi(\phi)$$

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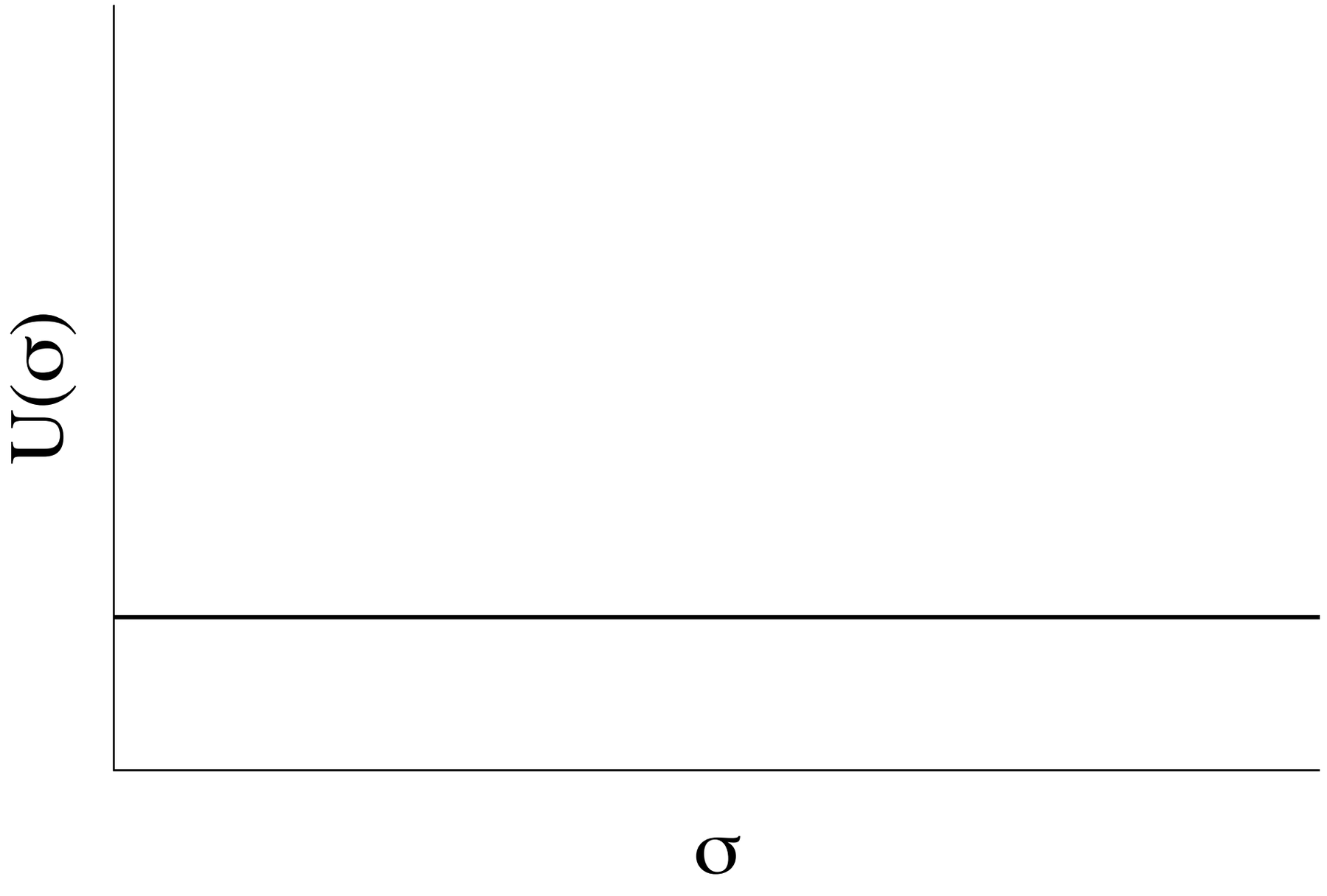
$$\prod_{n=1}^{N} \pi(y_n|\theta_n)\,\mathcal{N}(\theta_n|\mu,\sigma)\,\pi(\mu)\,\pi(\sigma)$$

Informative prior distributions on the population hyperparameters are incredibly important.

$$\pi(\mu)\,\pi(\sigma)$$

A uniform prior on the population deviance places too much probability at high values and no pooling.



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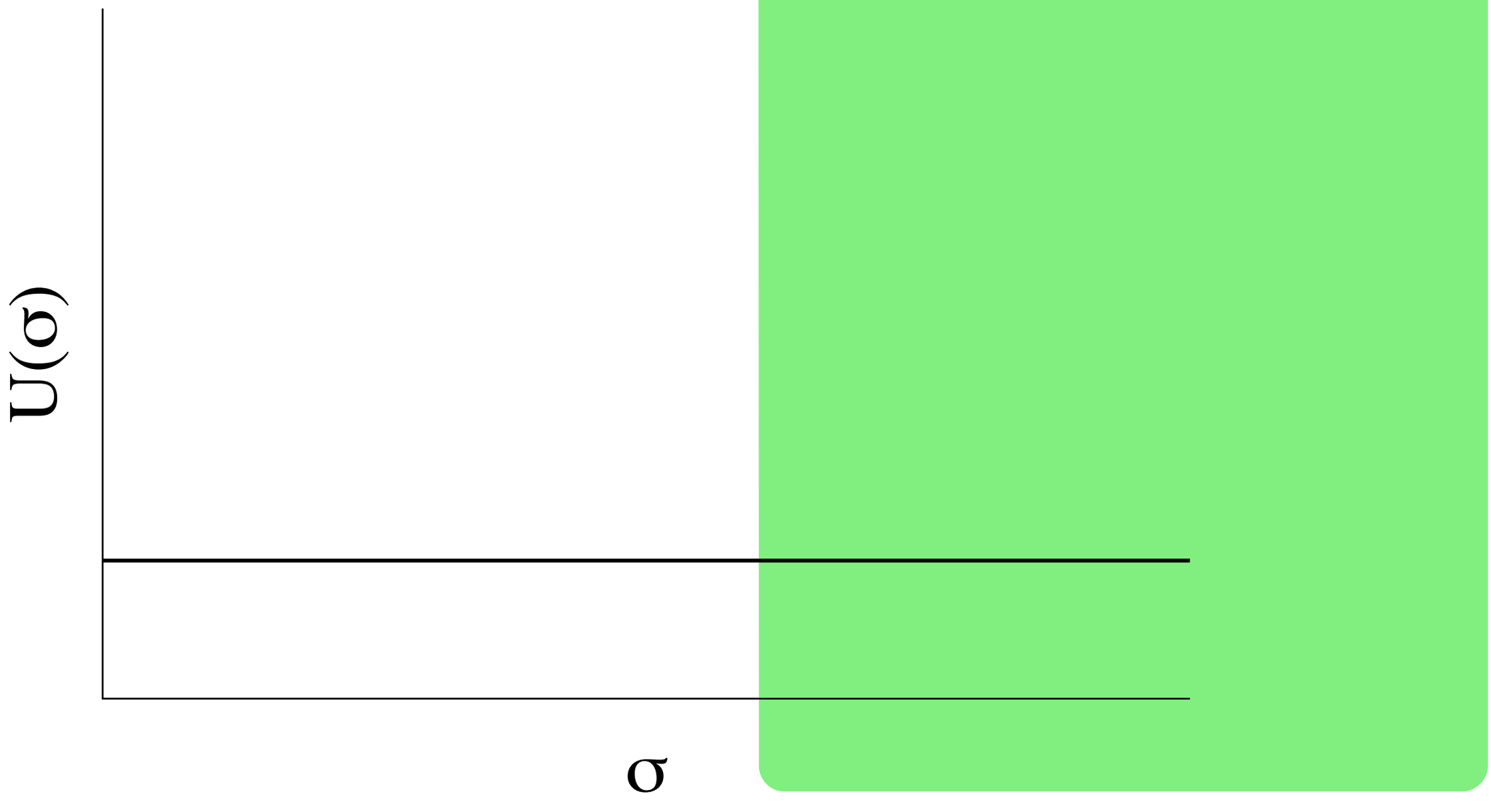

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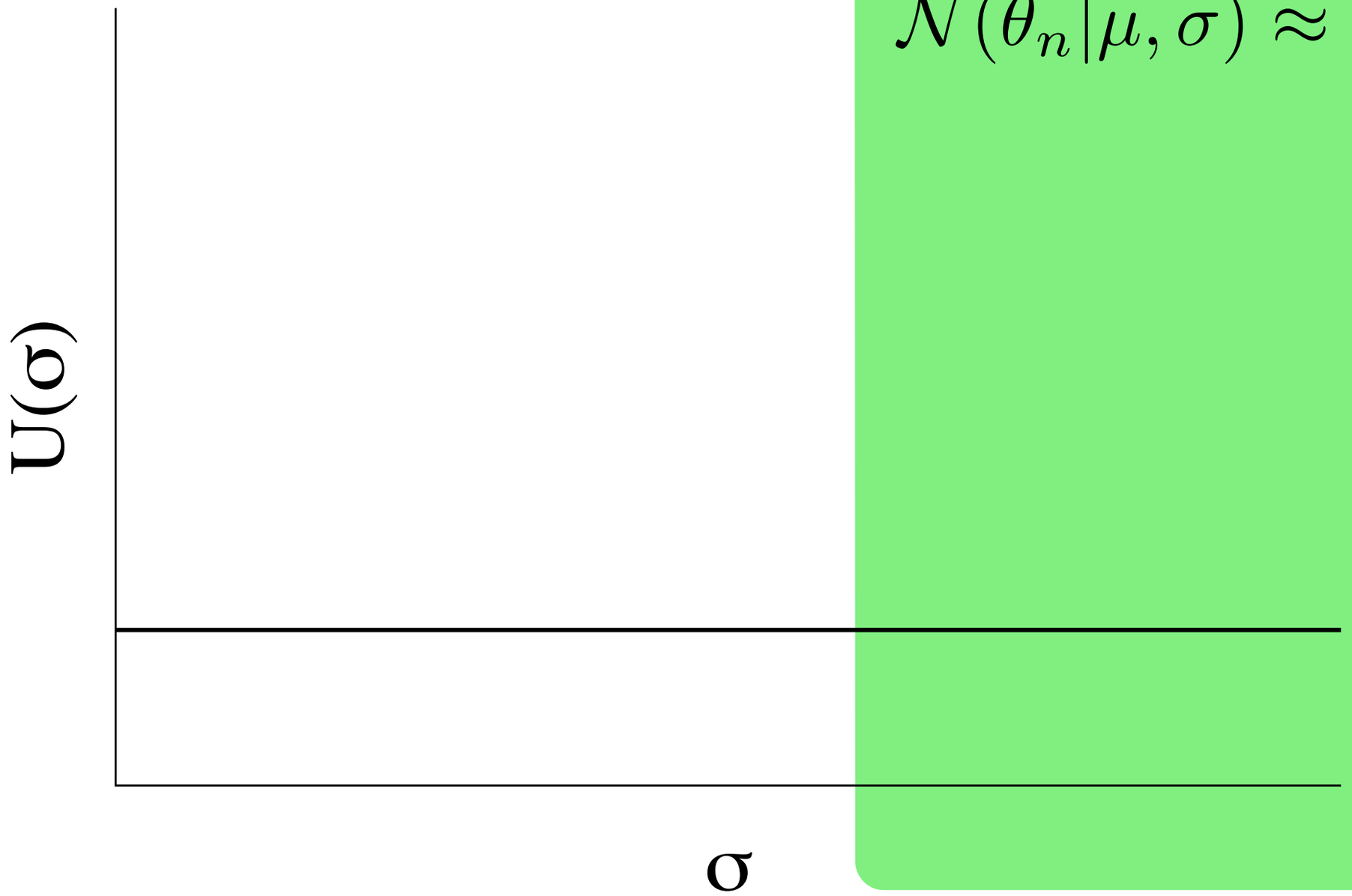

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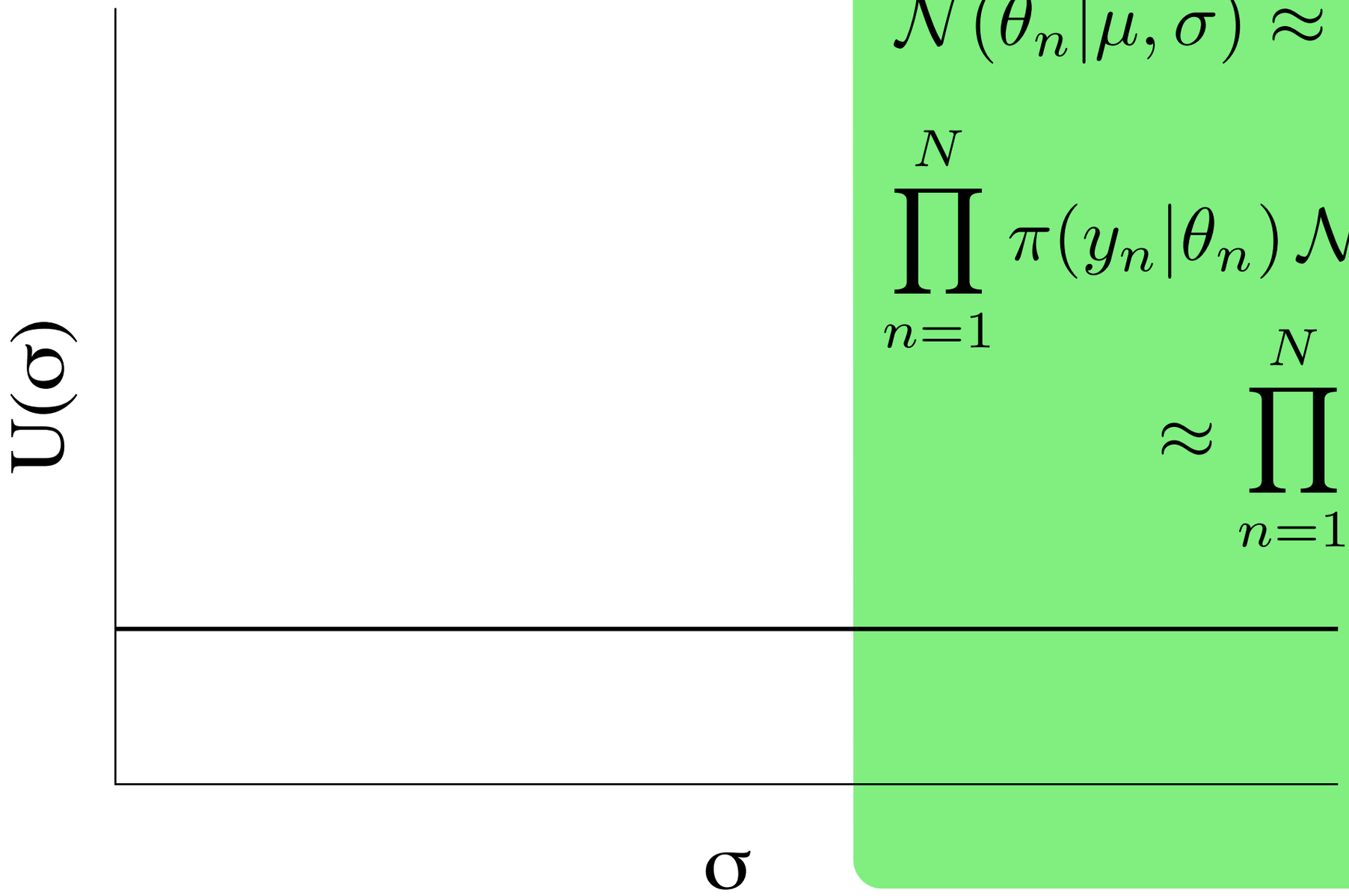

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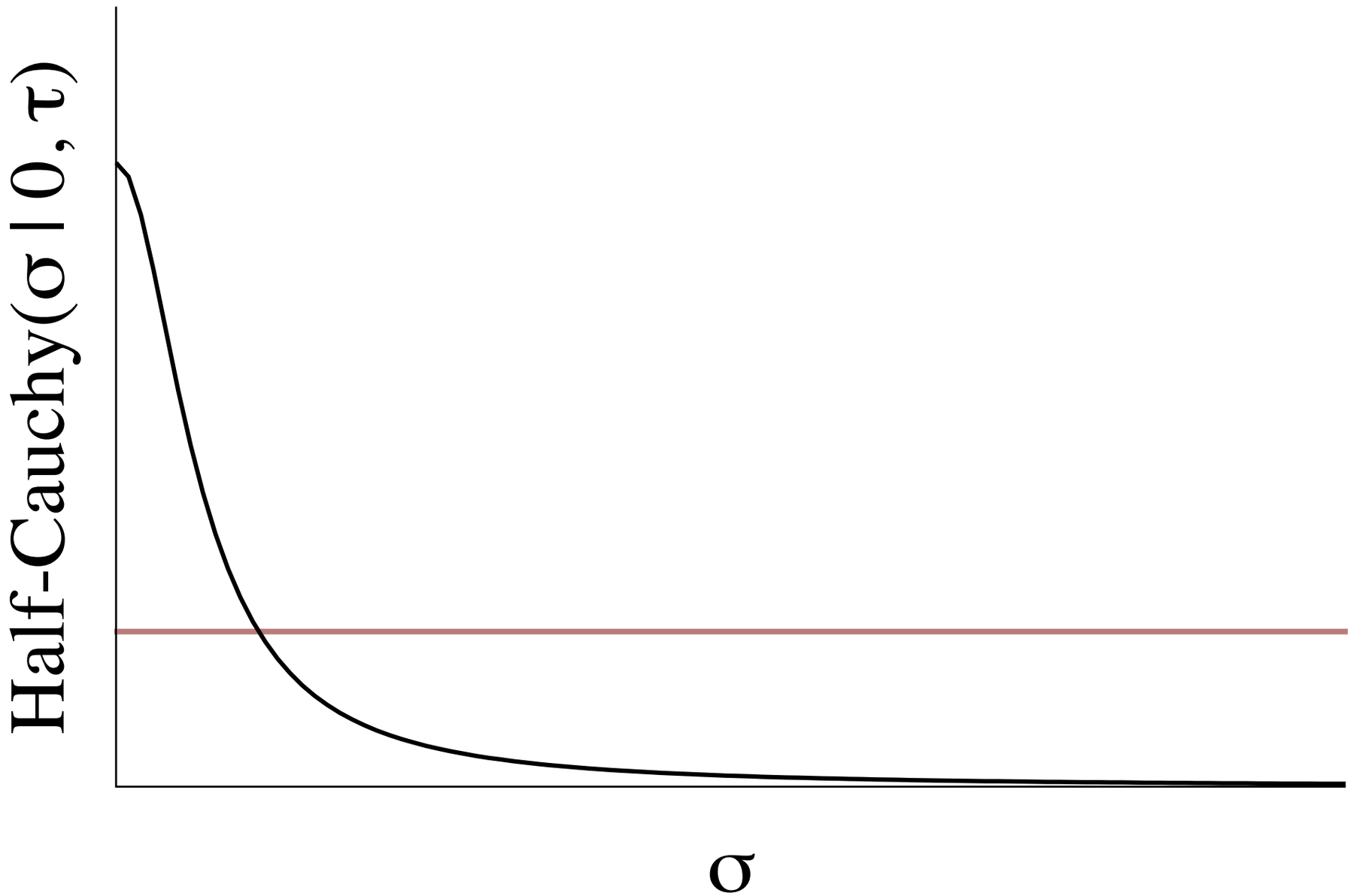
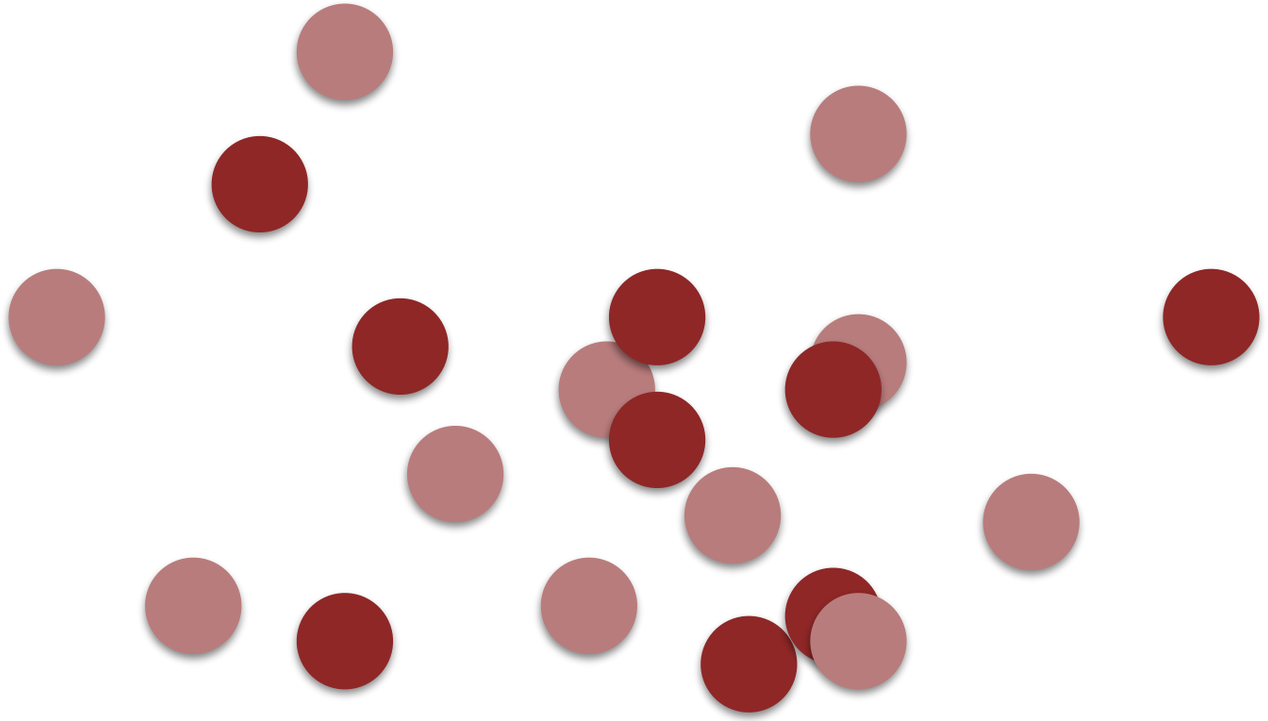
For the model to be able to partially pool we need a weakly informative prior that concentrates around zero.



General Linear Hierarchical Models



General linear models are naturally suited to exchangeability and hierarchical models.

$$\pi\left(y|g\left(\mathbf{X}^T\boldsymbol{\beta}+\alpha\right),\theta\right)$$

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$$\boldsymbol{\beta}_n \sim \mathcal{N}\left(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta\right)$$

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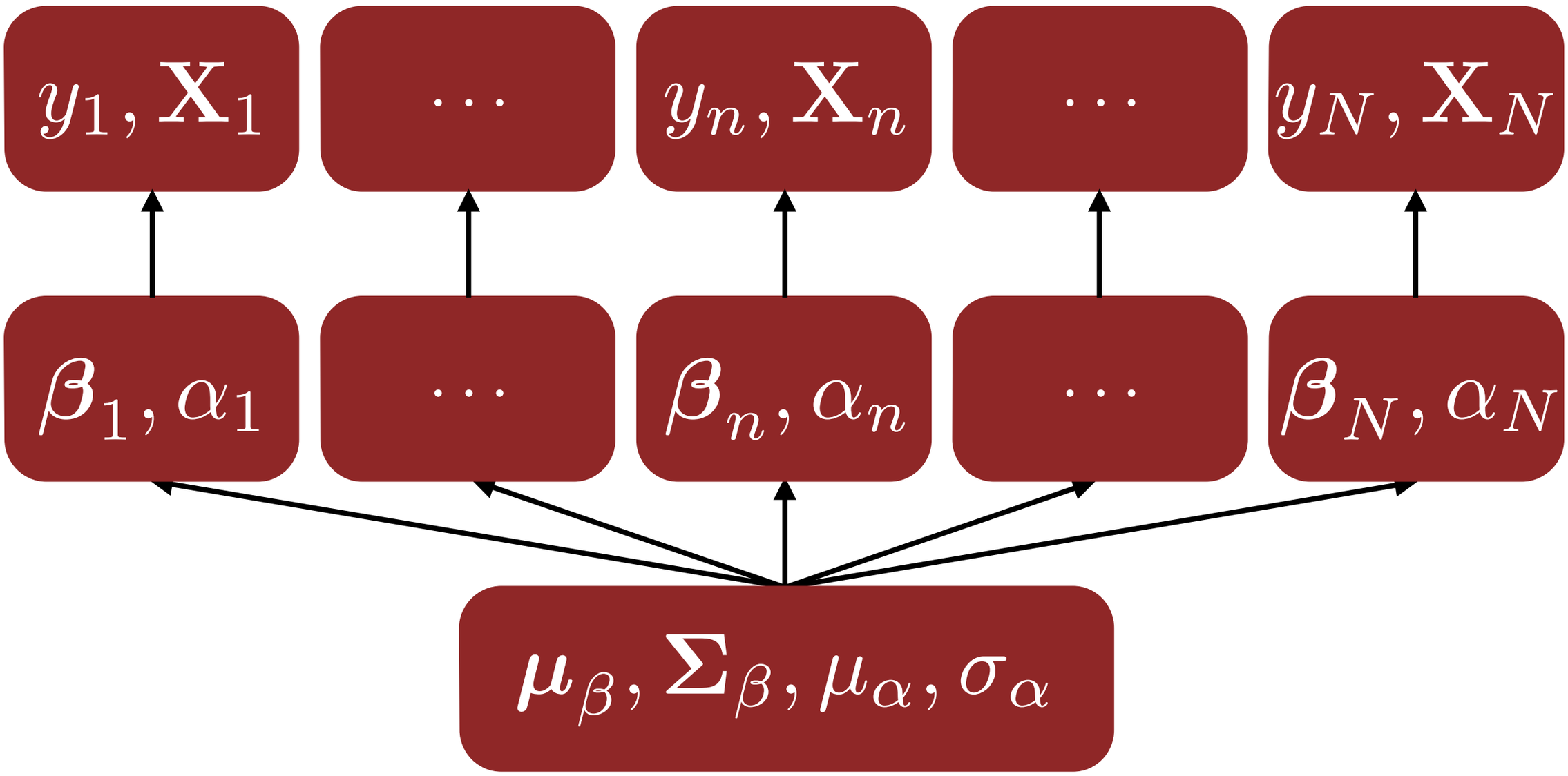

In a *multilevel* model the linear parameters vary across multiple hierarchies.

$$\pi\left(y_n | g\left(\mathbf{X}_n^T \boldsymbol{\beta}_n + \alpha_n\right), \theta\right)$$

$$\alpha_n = \sum_{k=1}^{N_{\text{groups}}} \alpha_{k,j(n)}$$

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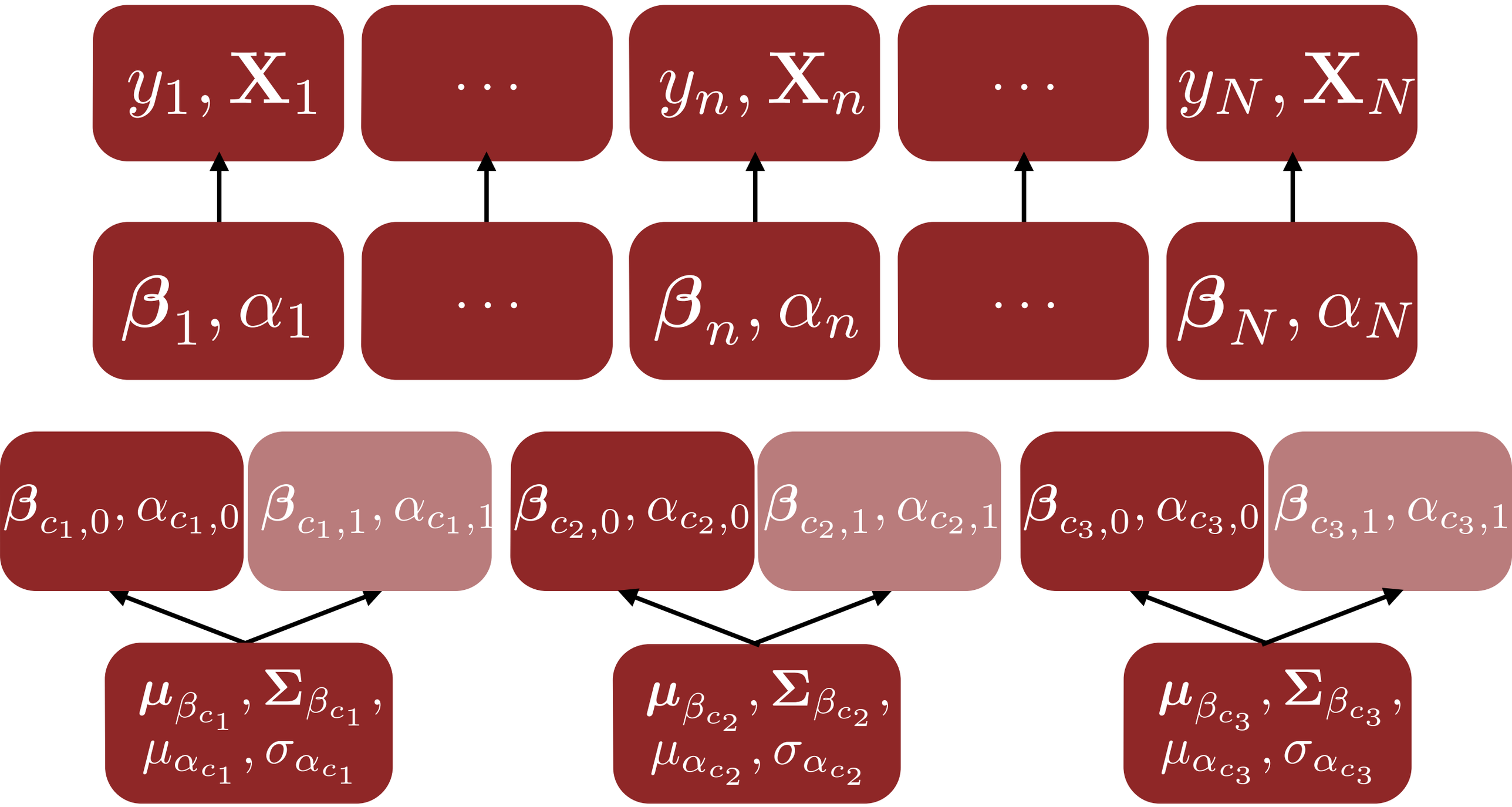

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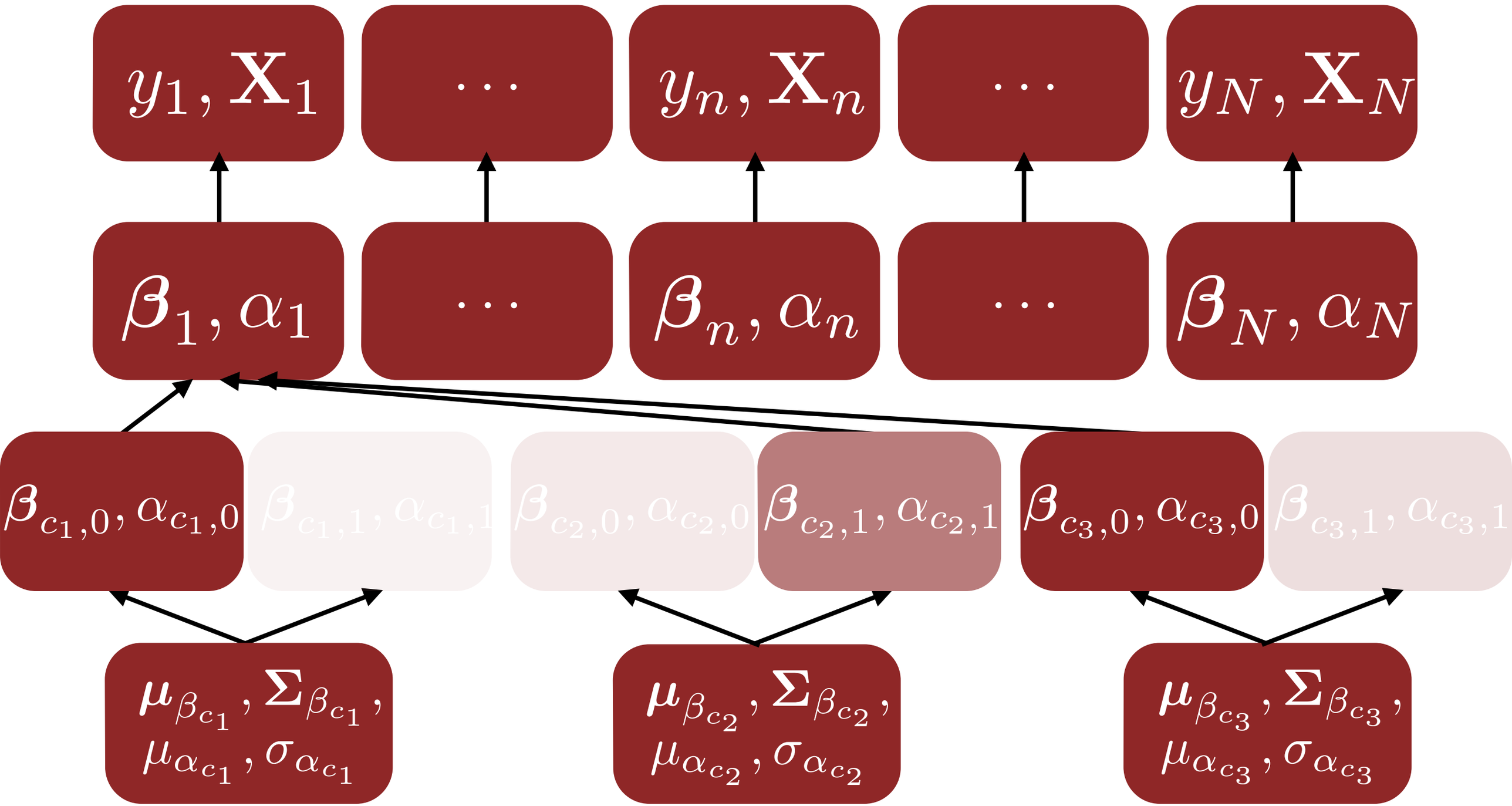

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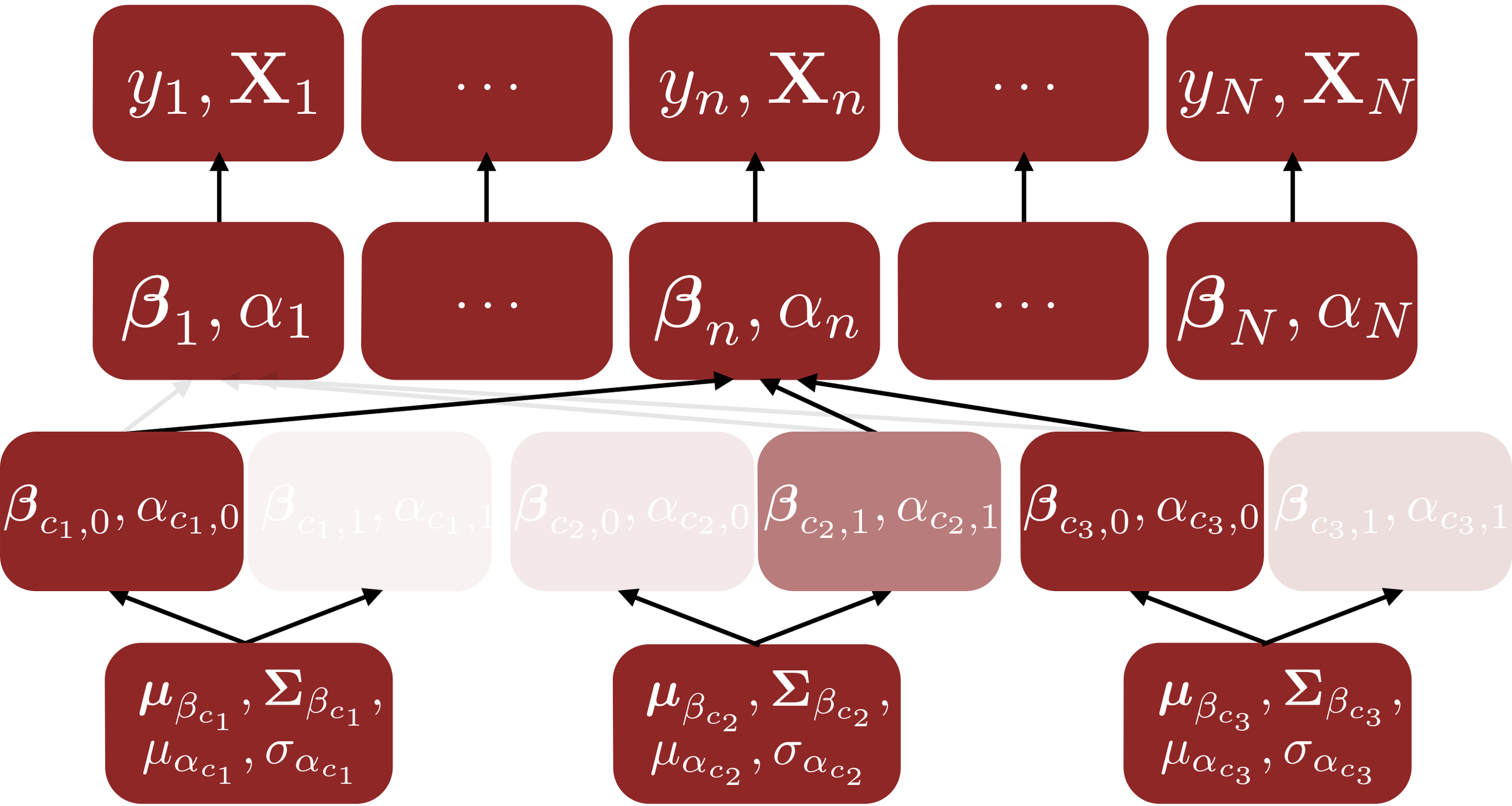

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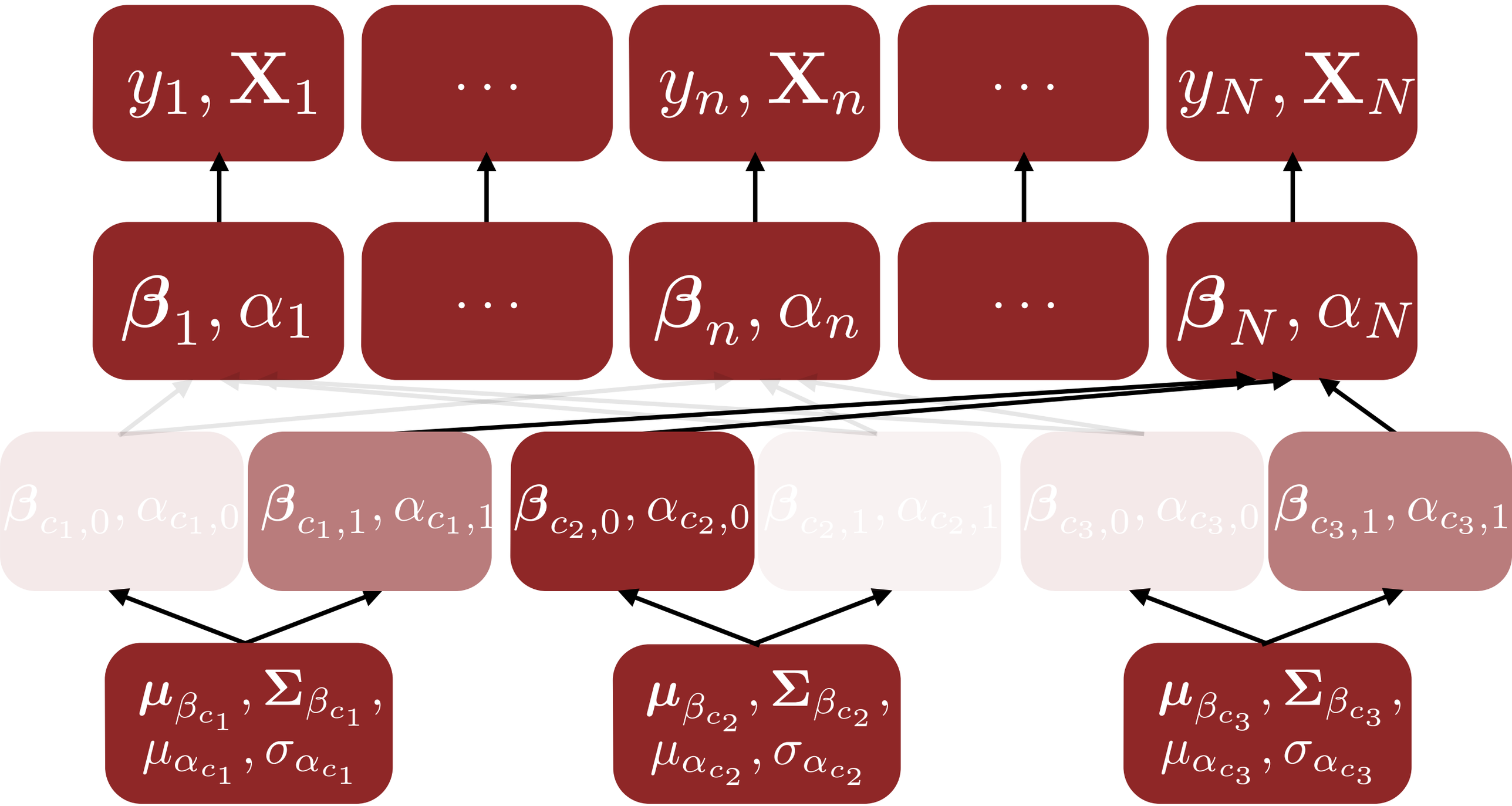
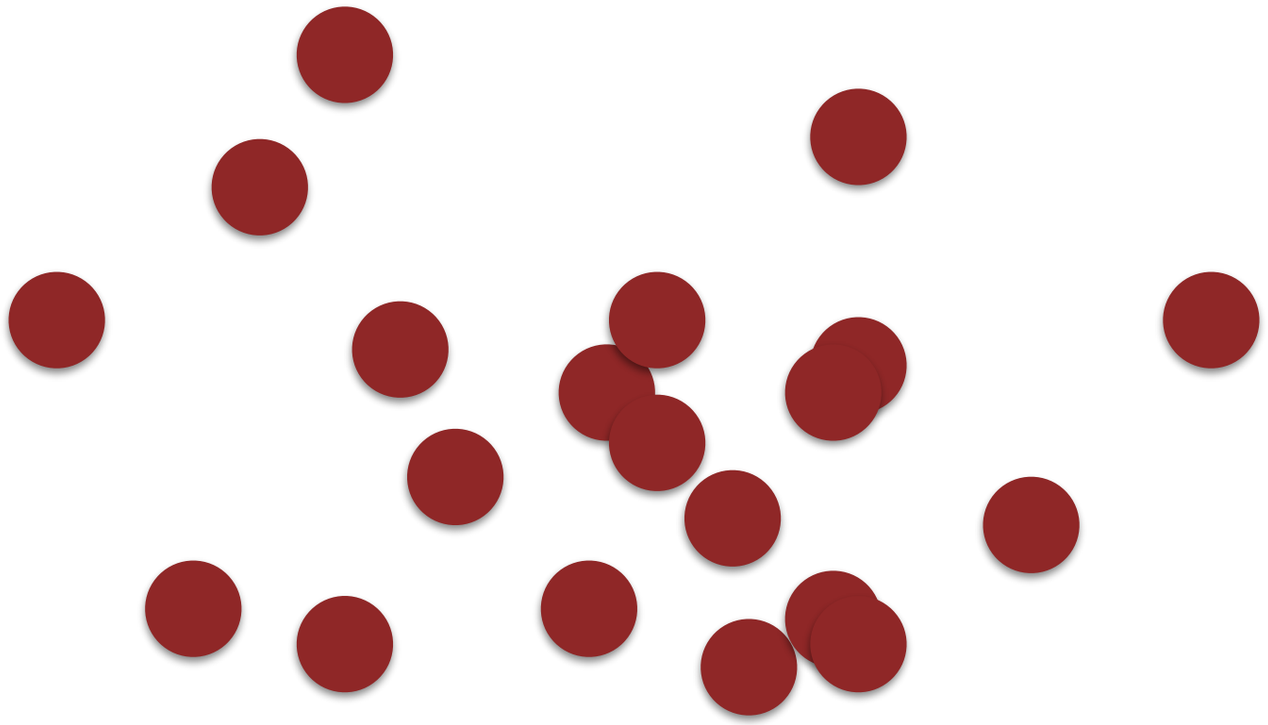


Each level contributes its own slope and intercept.



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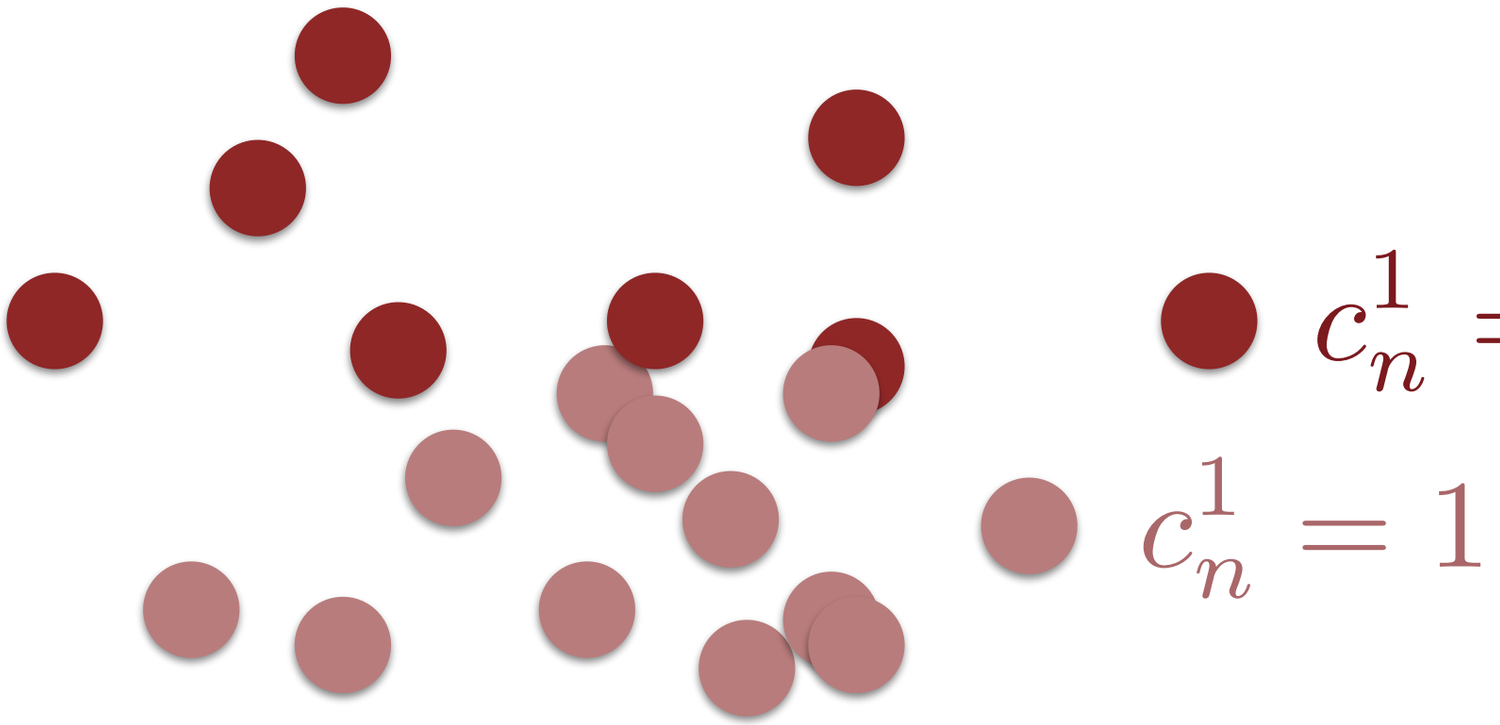

$$\alpha_{c_1,0} \quad \alpha_{c_1,1}$$

$$\alpha_{c_1,j} \sim \mathcal{N}\left(\mu_{\alpha_{c_1}}, \sigma_{\alpha_{c_1}}\right)$$

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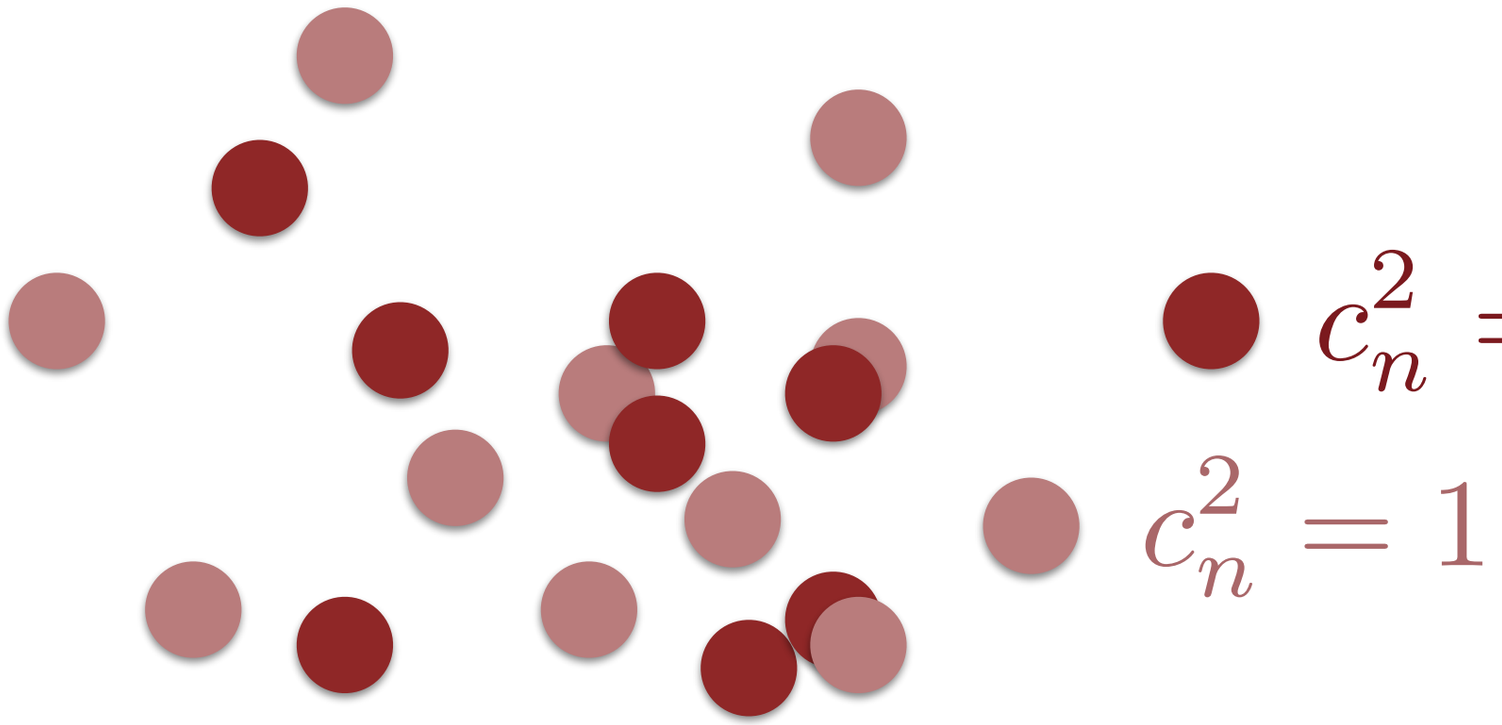

$$\alpha_{c_2,0} \quad \alpha_{c_2,1}$$

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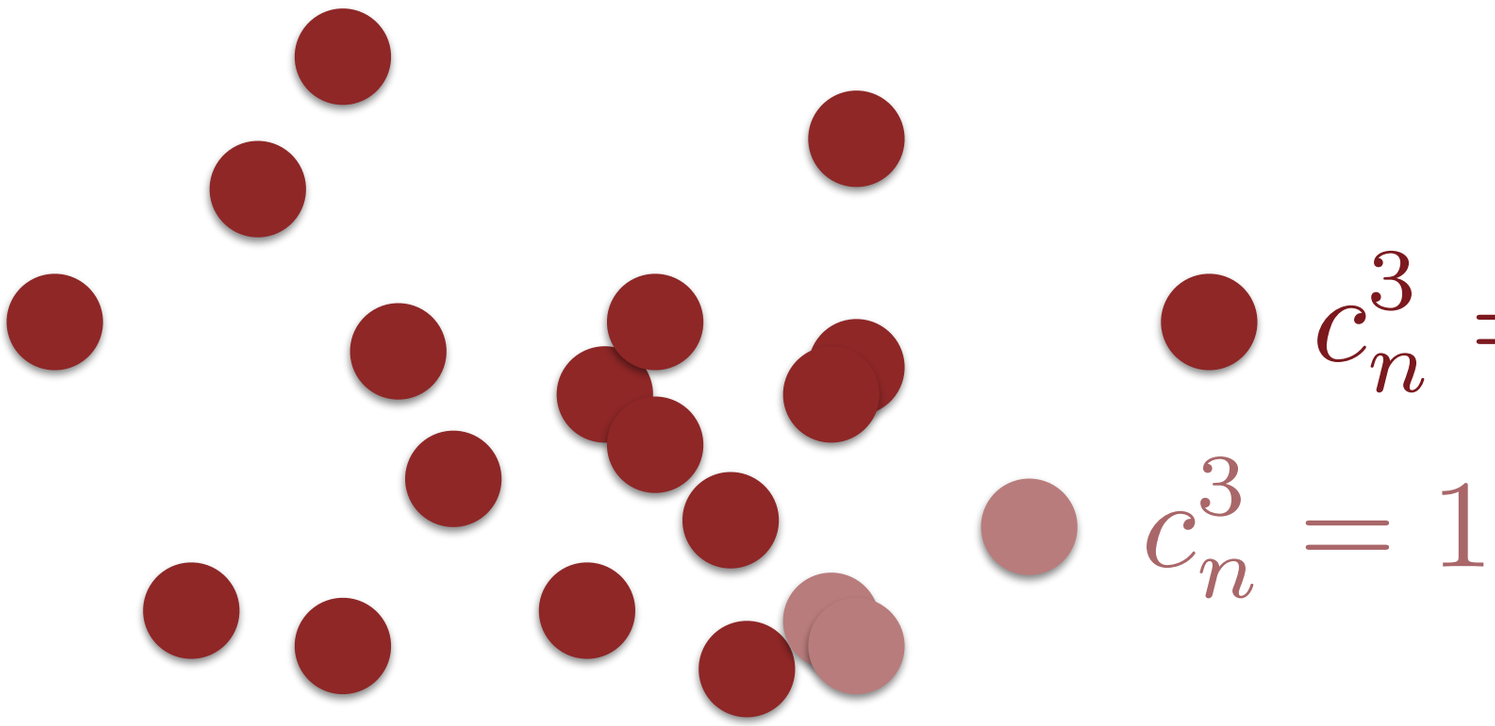

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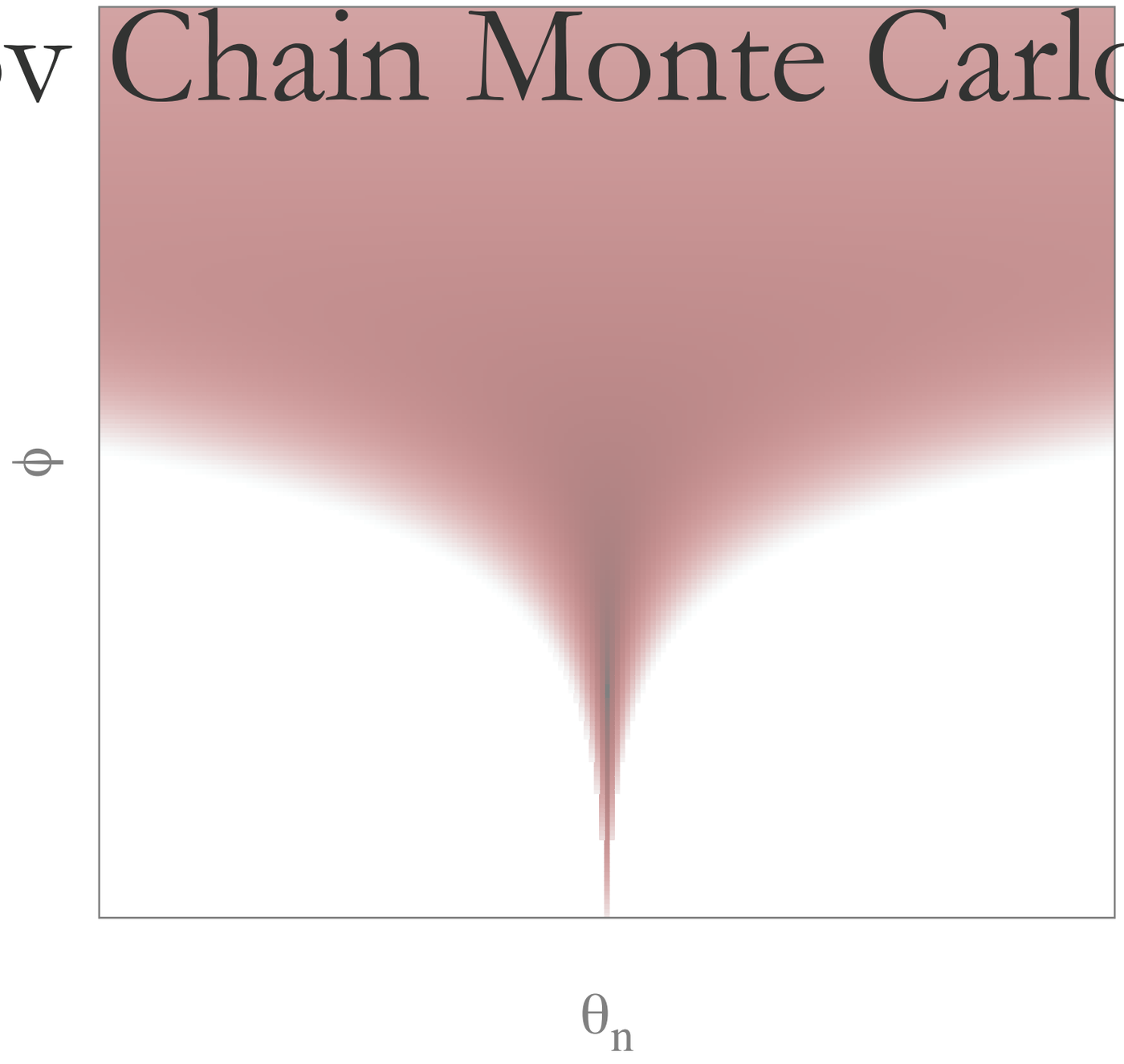
General Linear Hierarchical Models and Markov Chain Monte Carlo



Neal's funnel distribution illustrates some of the problems with sampling from hierarchical distributions.

$$\pi(\boldsymbol{\theta}|\phi) = \prod_{n=1}^{N} \mathcal{N}\left(\theta_n|0, e^{\frac{\phi}{2}}\right) \mathcal{N}(\phi|0,3)$$

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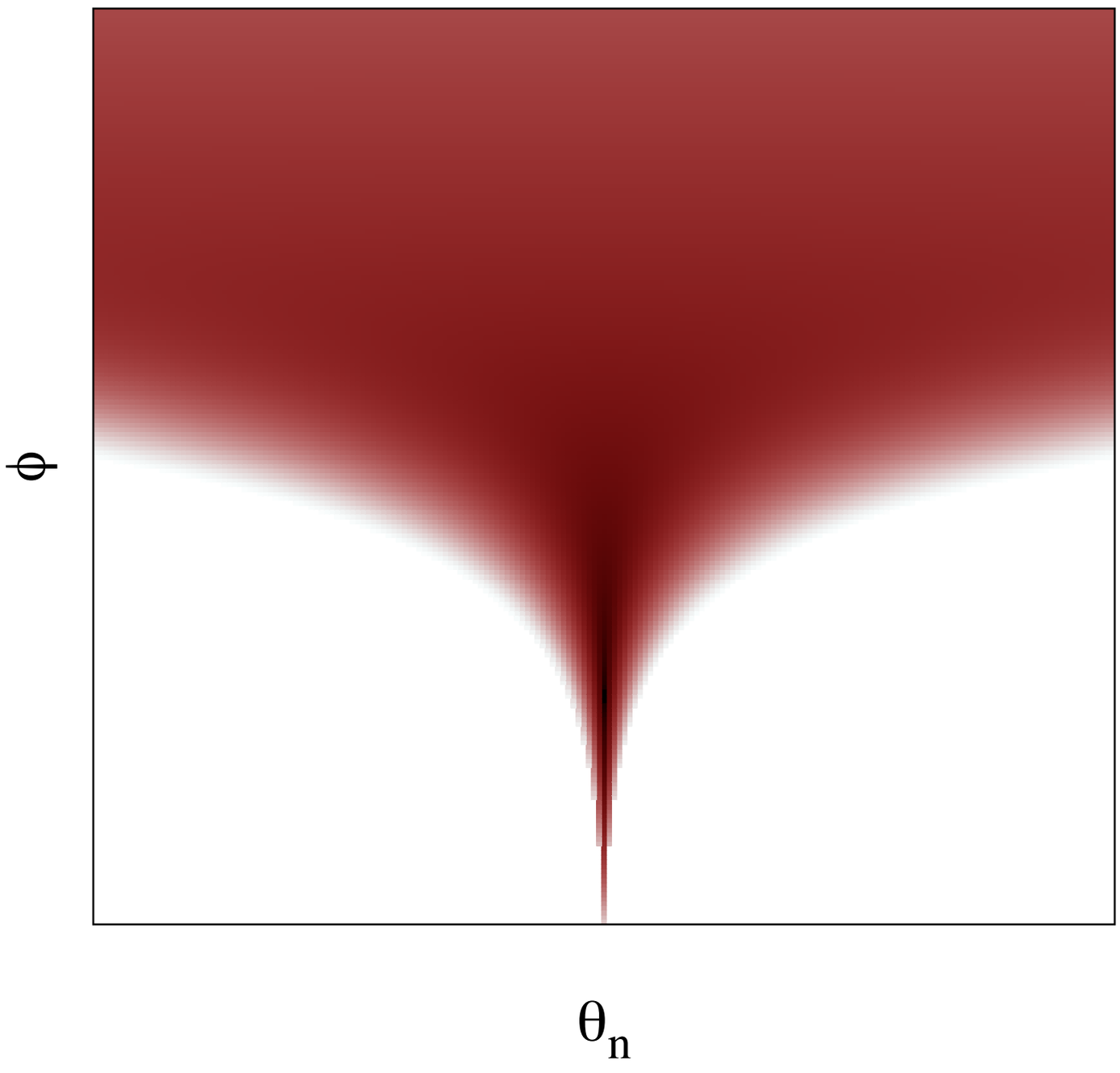

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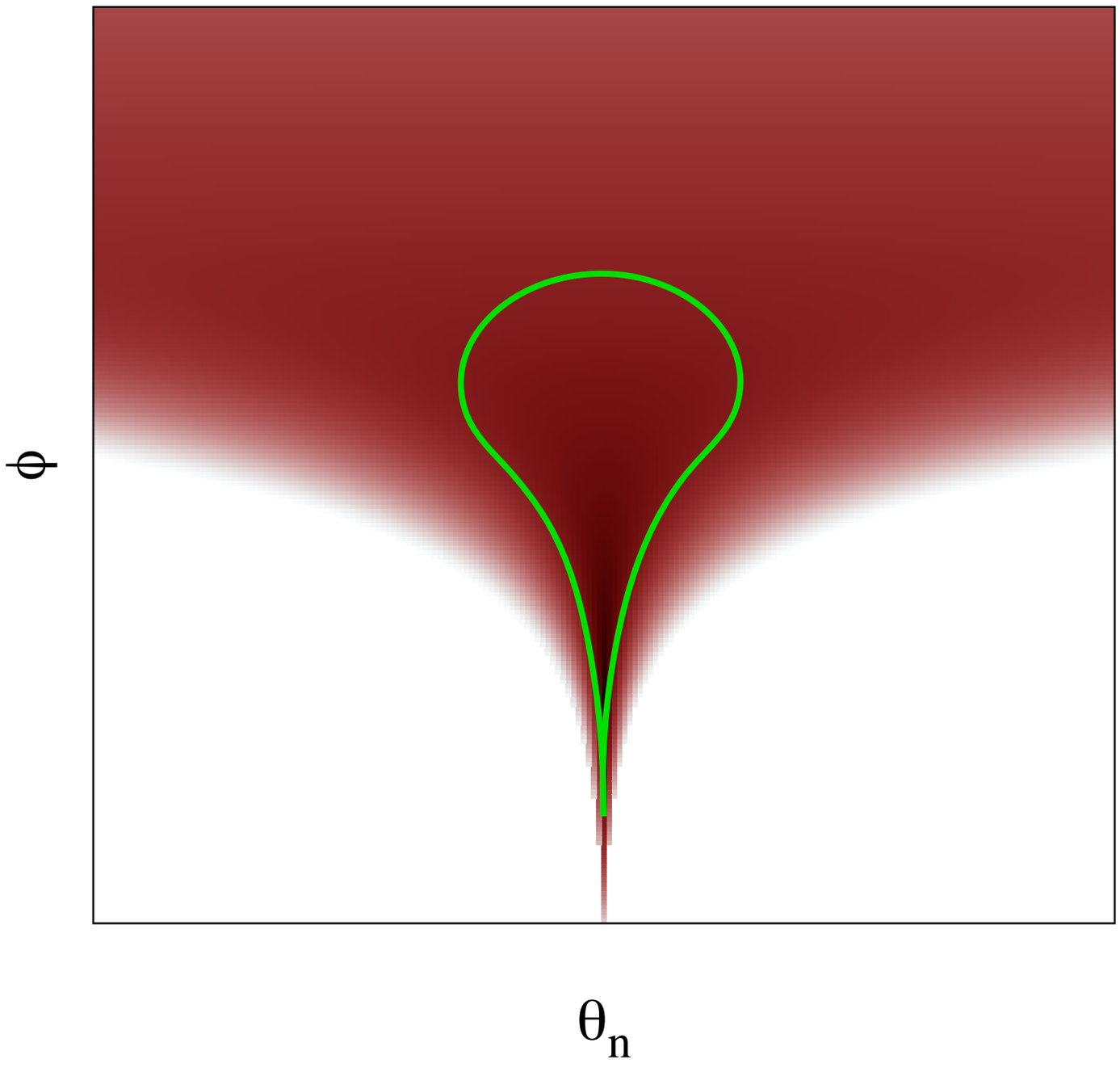

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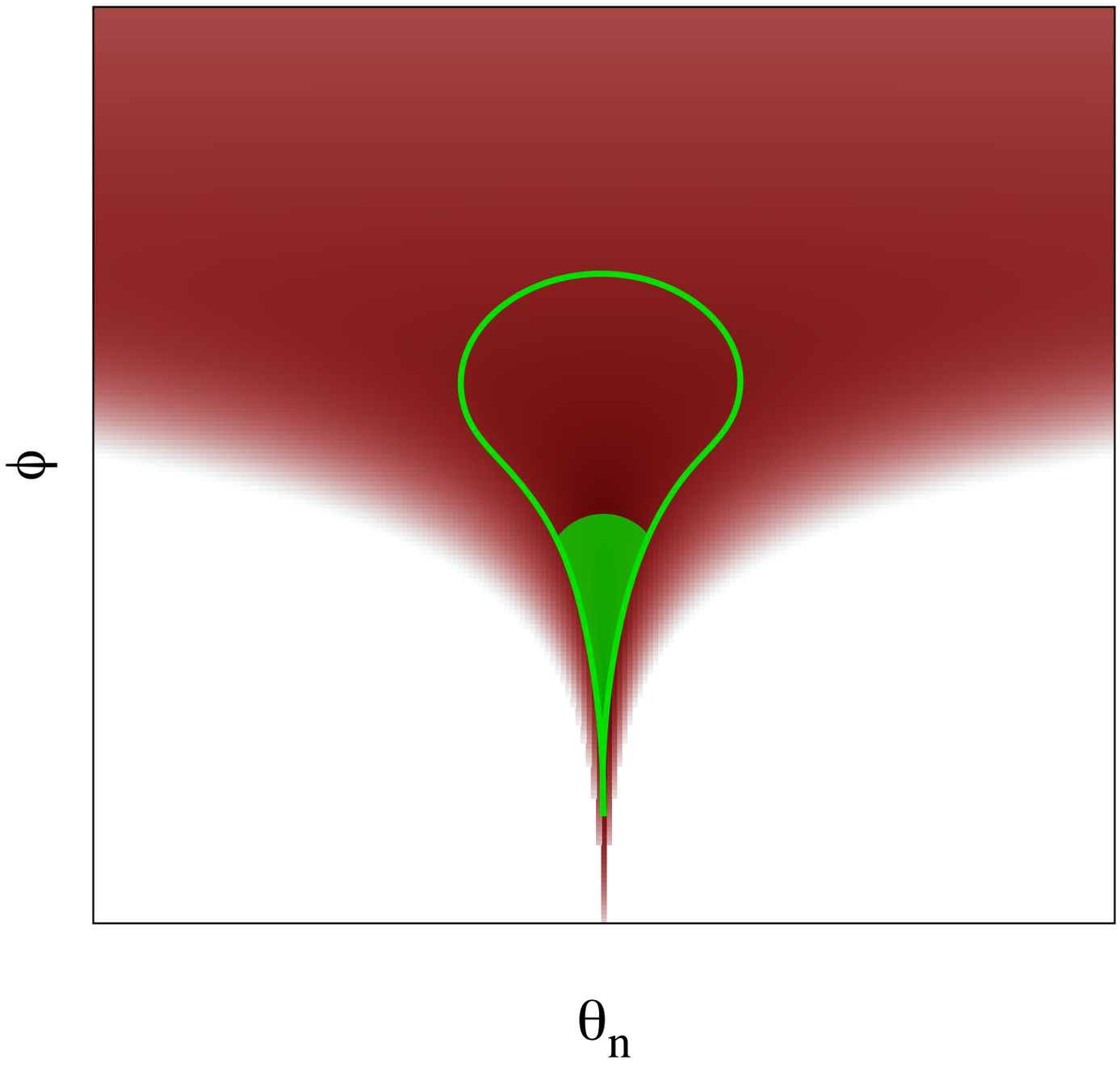

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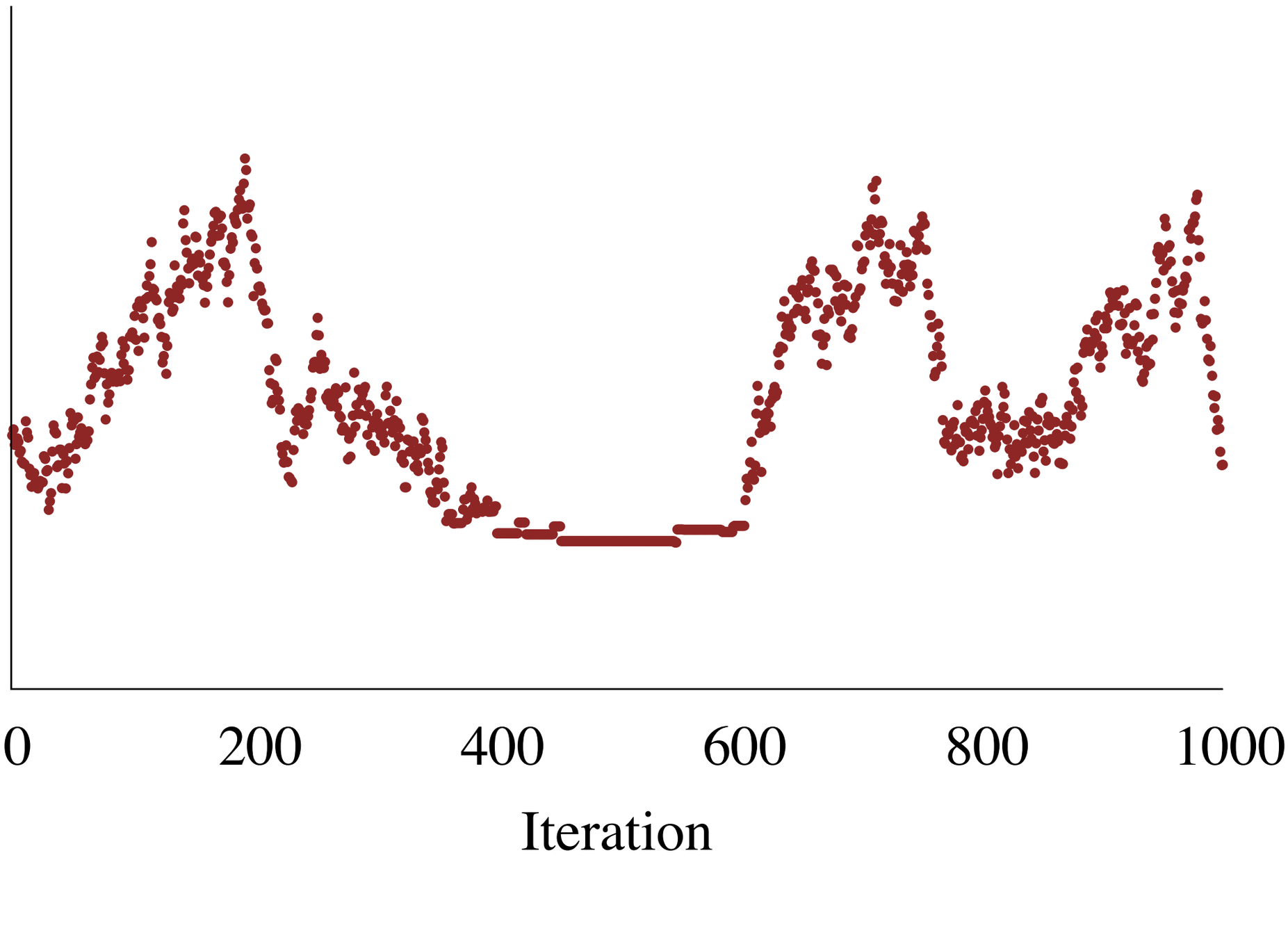

The geometry of the funnel, however, dramatically improves when we sample from auxiliary parameters.

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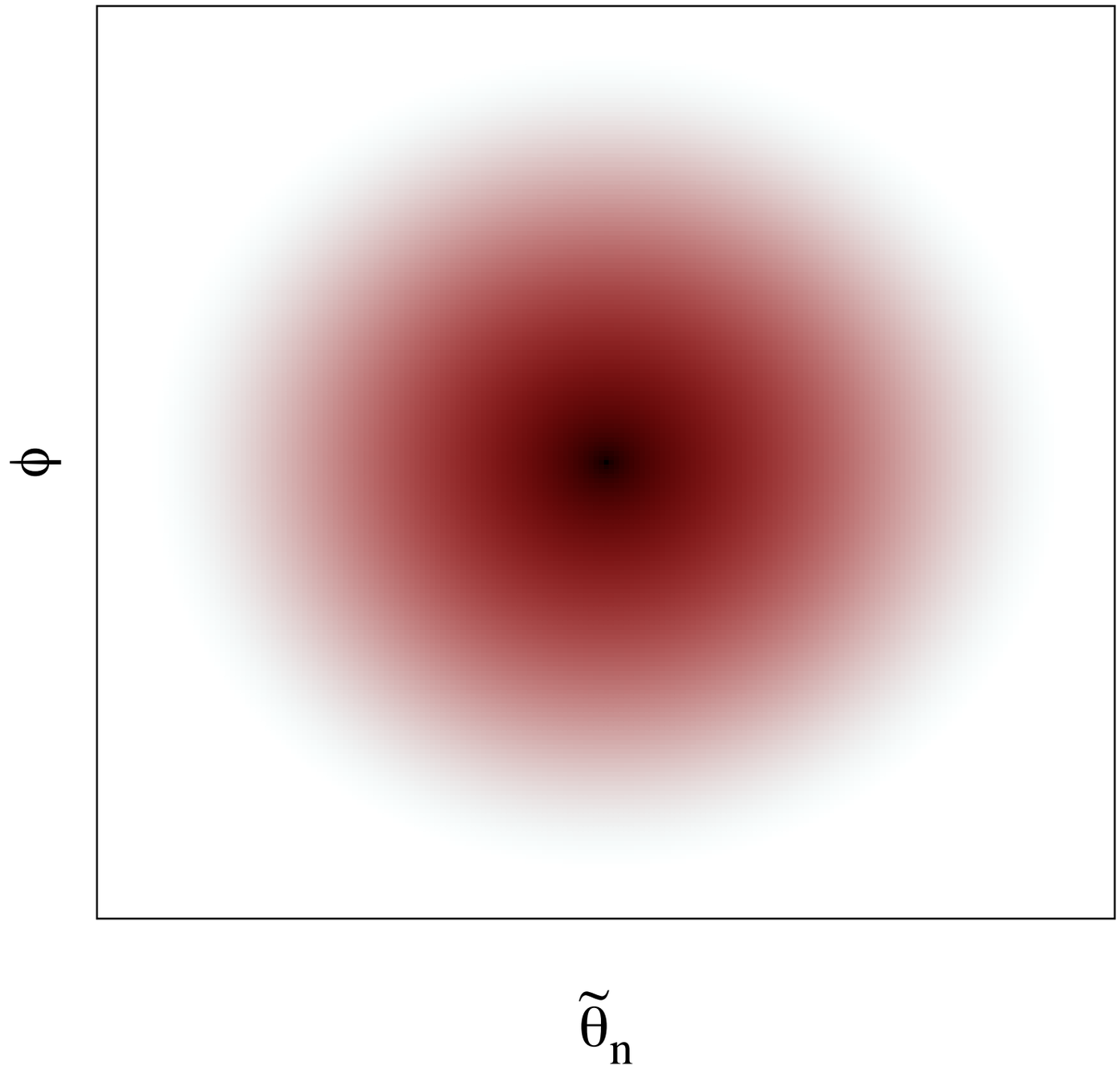


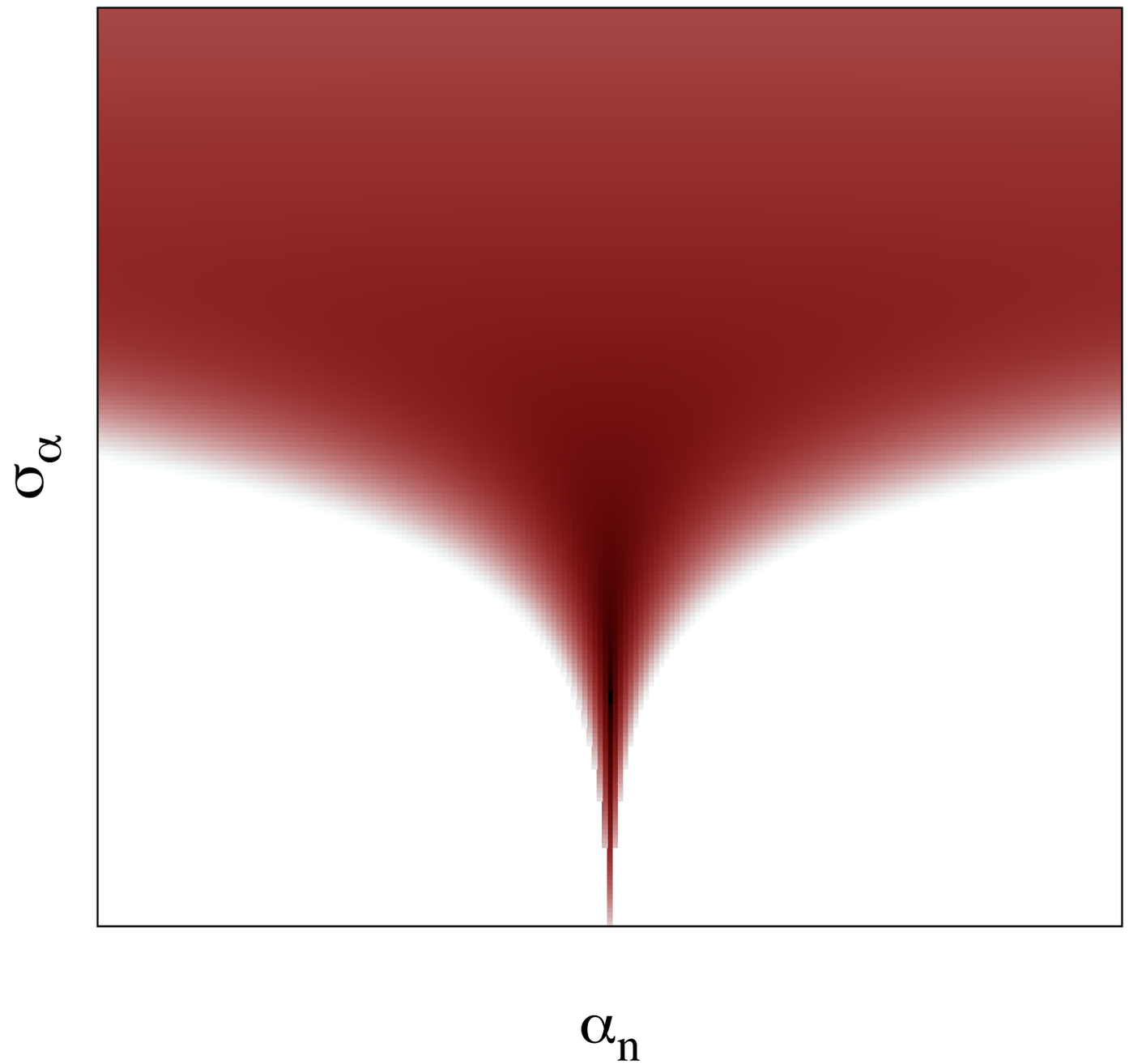
The most natural representation of a general linear hierarchical model is a *centered parameterization*.

$$\pi\left(y_n | g\left(\mathbf{X}_n^T \boldsymbol{\beta}_n + \alpha_n\right), \theta\right)$$

$$\boldsymbol{\beta}_n \sim \mathcal{N}\left(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta\right)$$

$$\alpha_n \sim \mathcal{N}\left(\mu_\alpha, \sigma_\alpha\right)$$

When data are sparse and uninformative, the centered parameterization of the posterior resembles the funnel.



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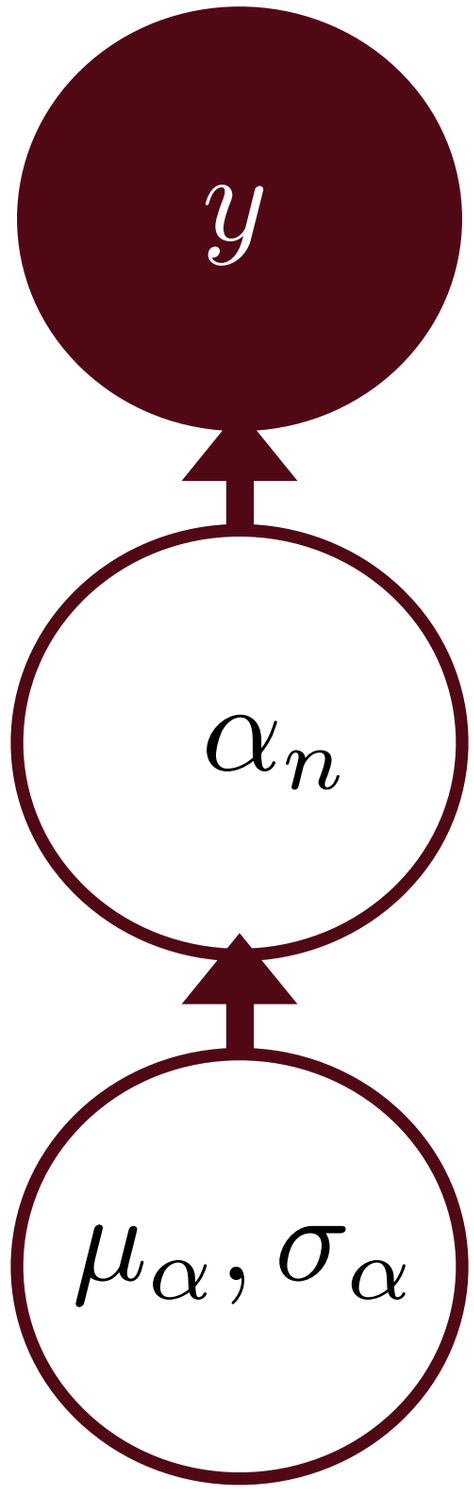
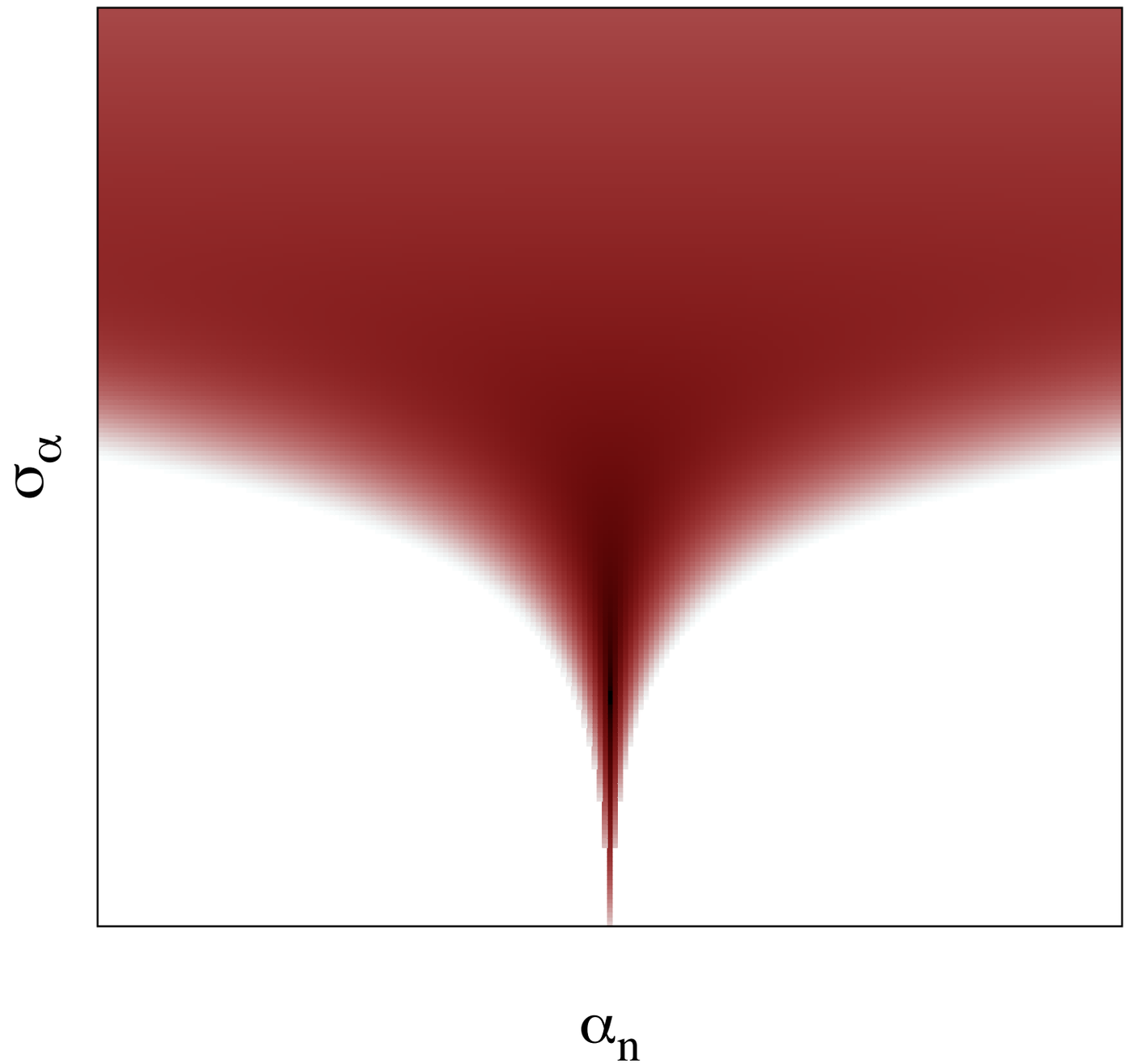


But when the data are more informative the posterior geometry is much nicer.



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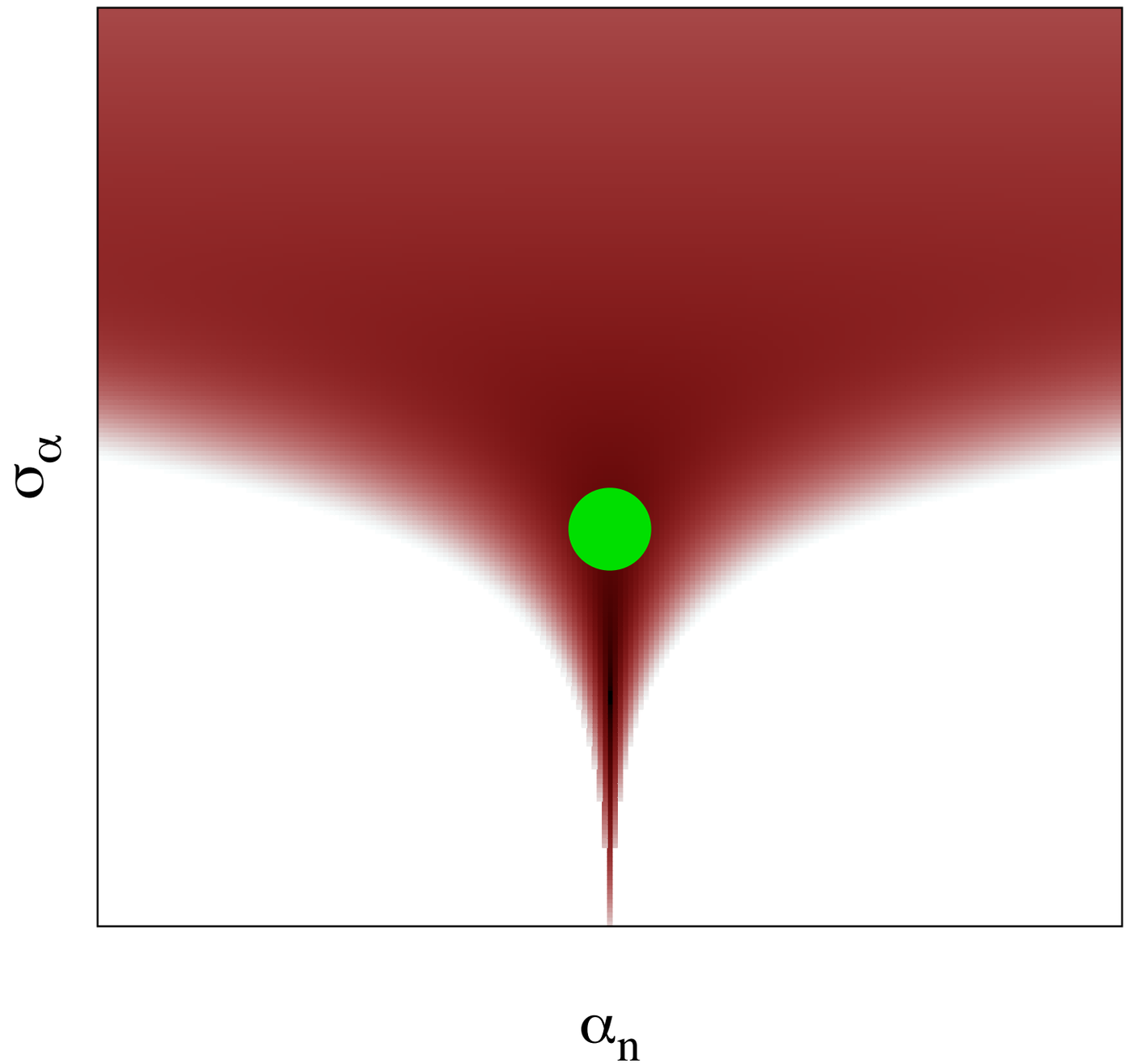

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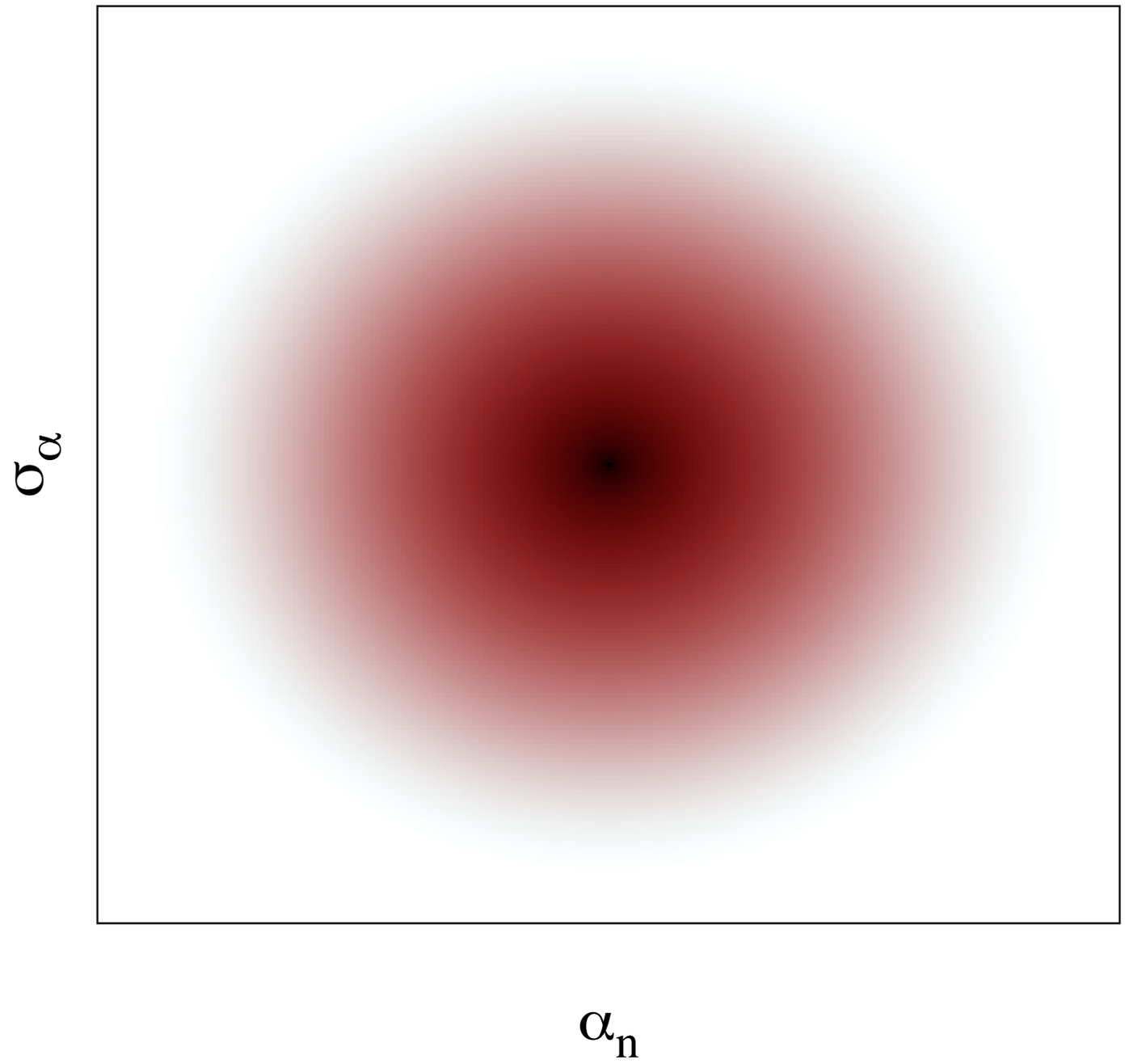


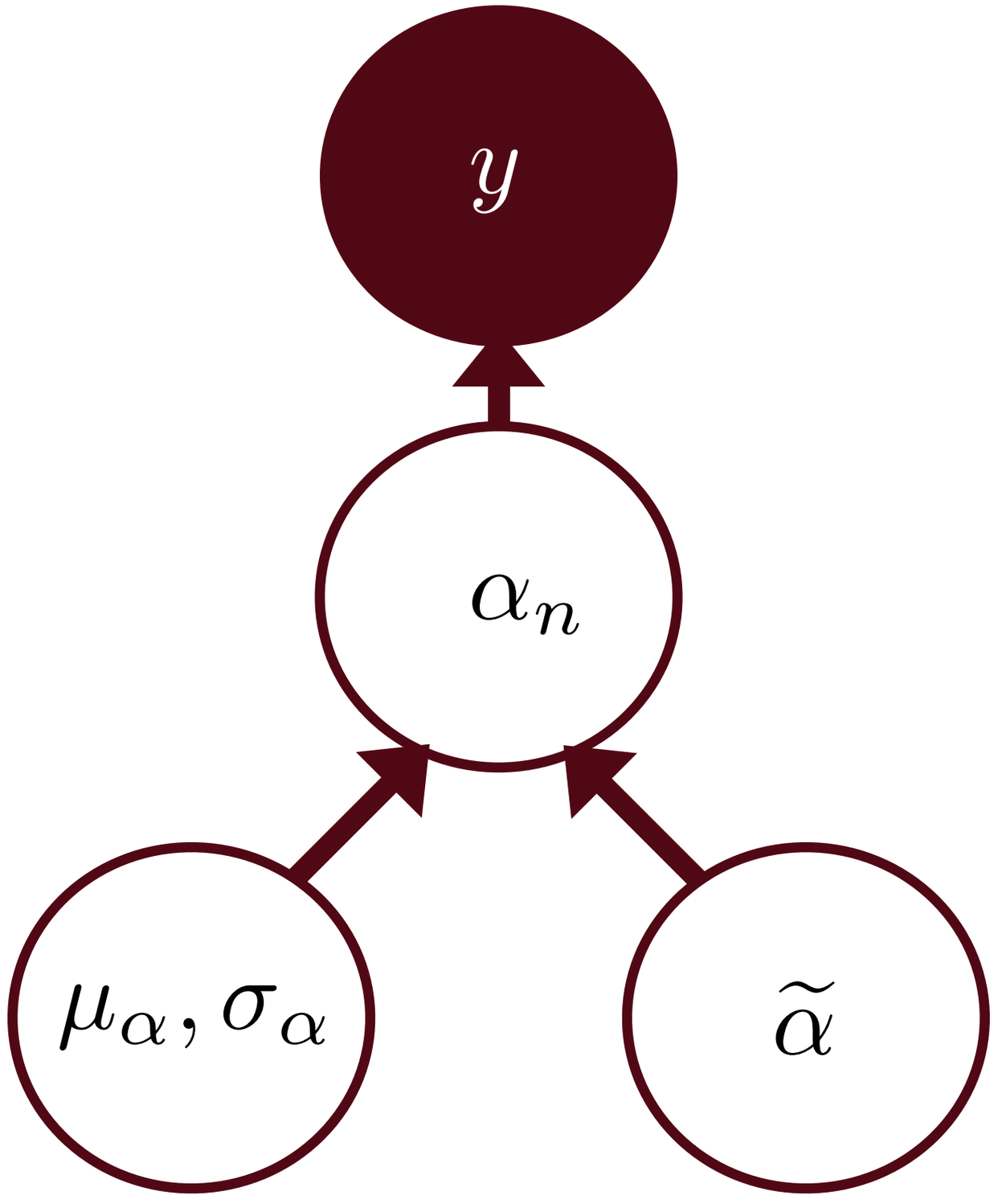
The same general linear hierarchical model can also be implemented with a *non-centered parameterization*.

$$\pi\left(y_n | g\left(\mathbf{X}_n^T \boldsymbol{\beta}_n + \alpha_n\right), \theta\right)$$

$$\tilde{\boldsymbol{\beta}}_n \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$$
$$\boldsymbol{\beta}_n = \boldsymbol{\mu}_\beta + \mathbf{L}\tilde{\boldsymbol{\beta}}_n,\ \mathbf{L}\mathbf{L}^T = \boldsymbol{\Sigma}_\beta$$

$$\tilde{\alpha}_n \sim \mathcal{N}(0, 1)$$
$$\alpha_n = \mu_\alpha + \sigma_\alpha \tilde{\alpha}_n$$

When data are sparse, the non-centered parameterization of the posterior yields a favorable geometry.



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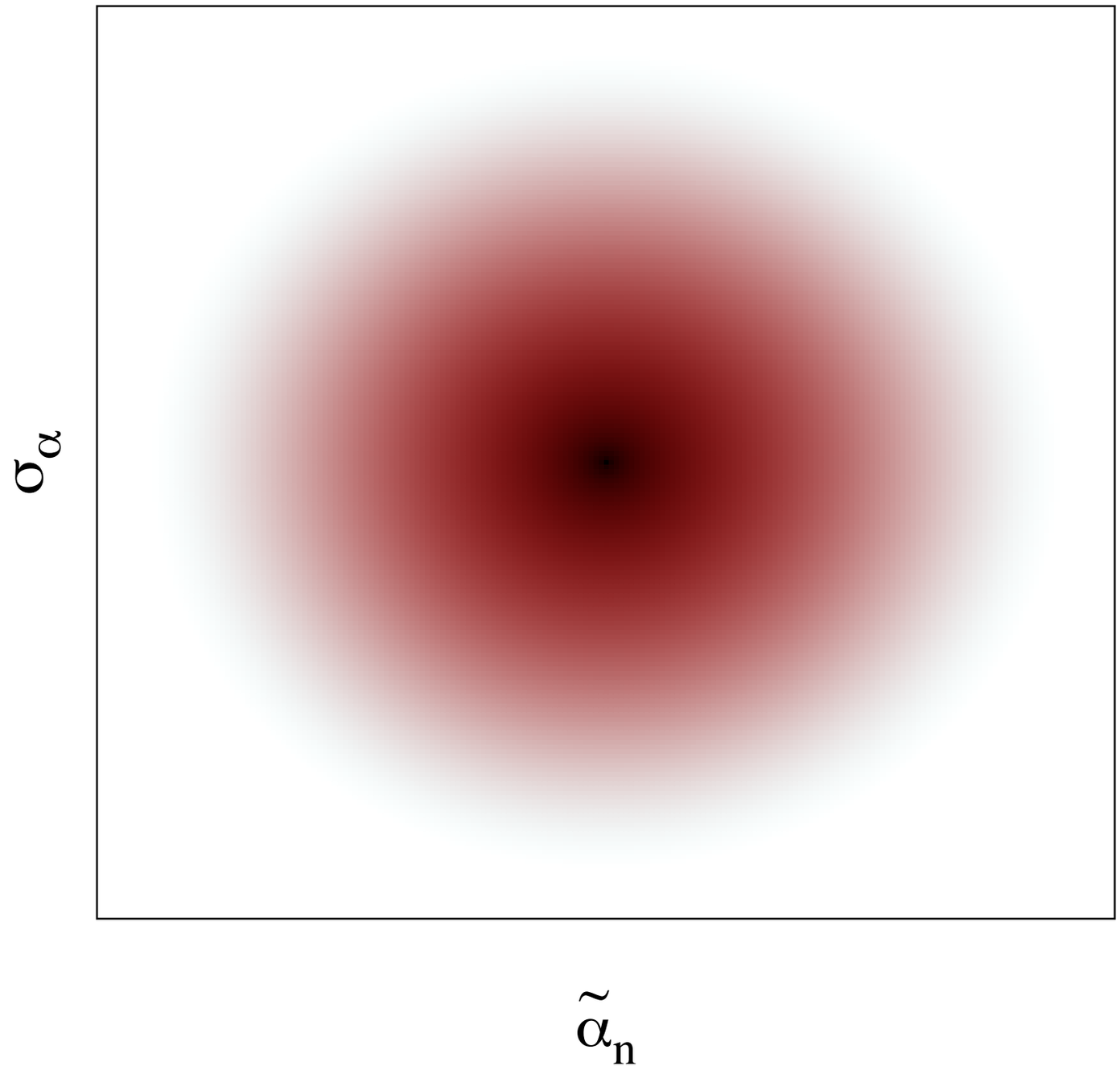
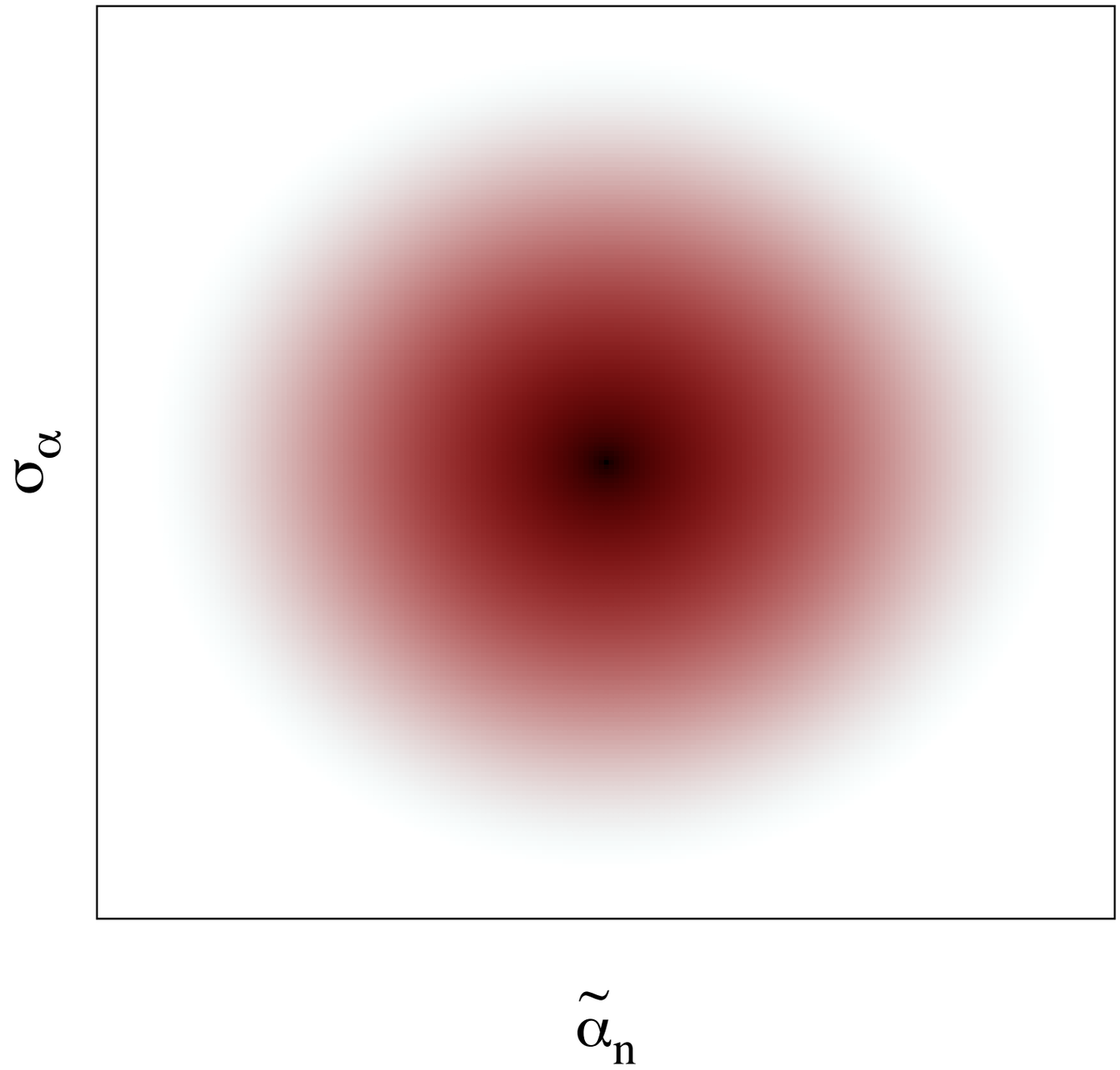


The constraints imposed by the data, however, are more complex, yielding pathological posteriors with more data.



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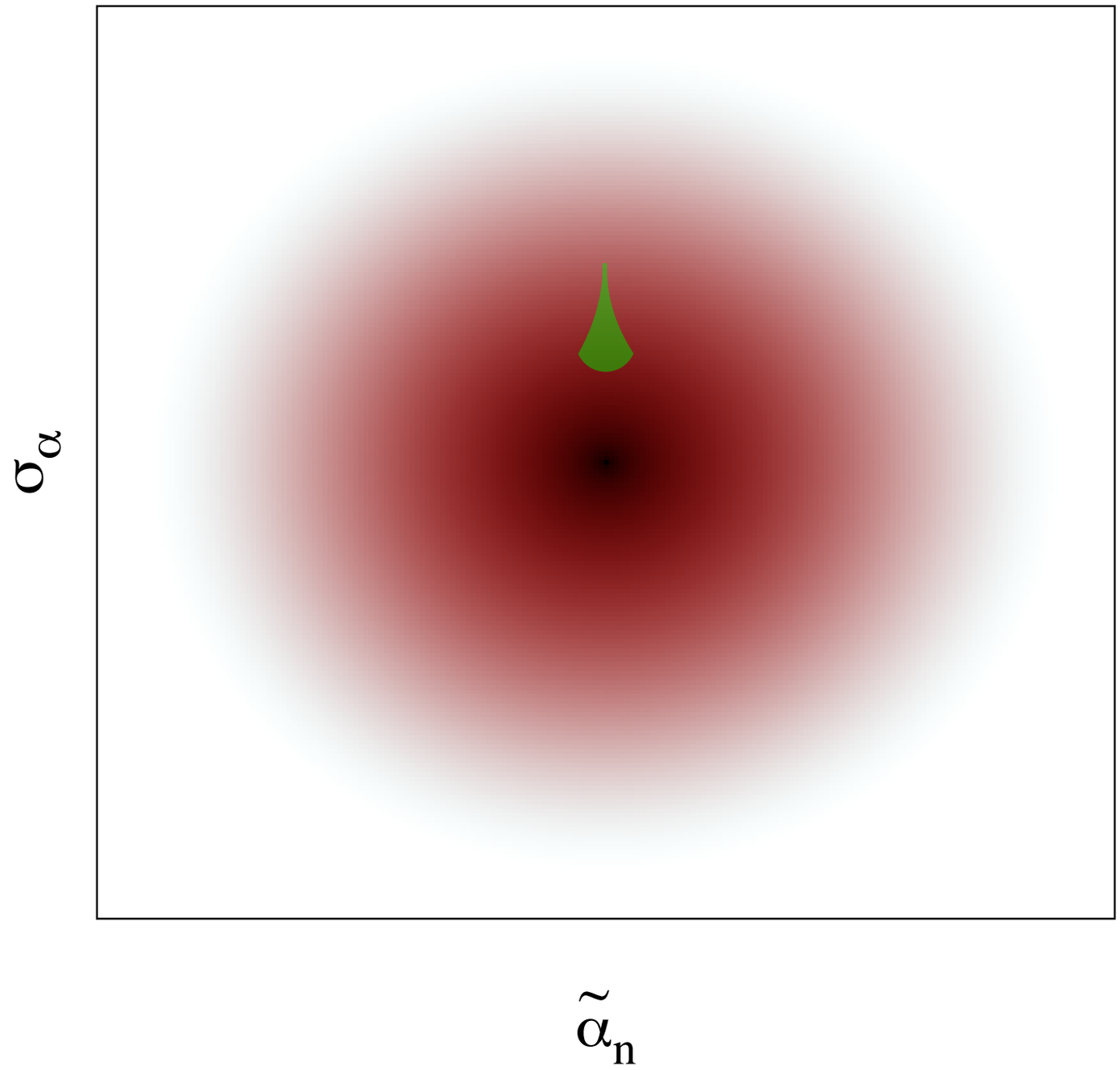

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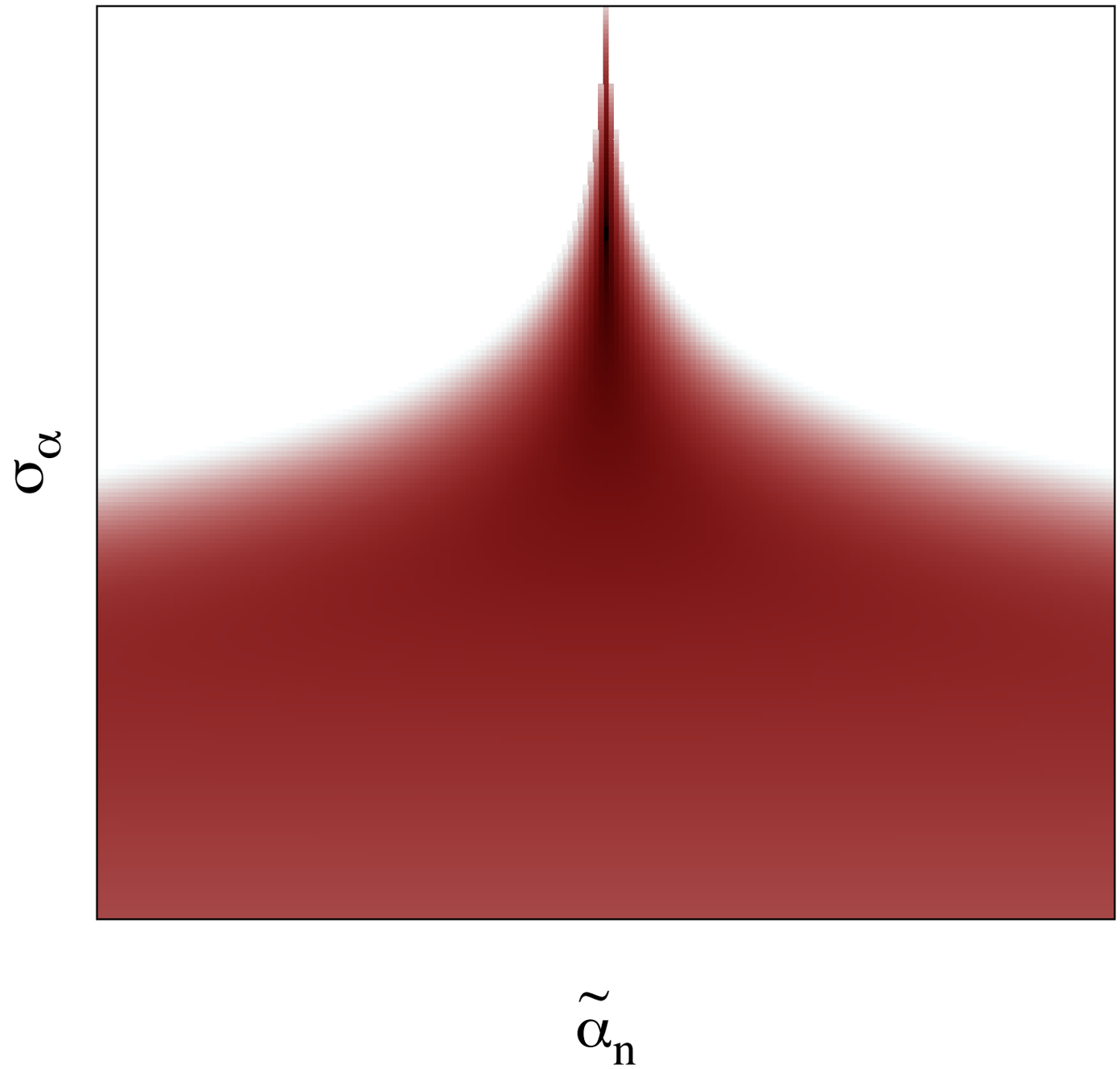

Consequently the two parameterizations are each advantageous in different circumstances.

Centered Parameterization

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Hierarchical
Geometry

Centered
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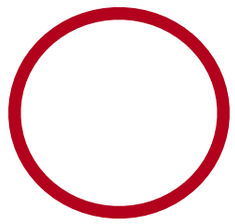
	Hierarchical Geometry	Measurement Geometry
Centered Parameterization		

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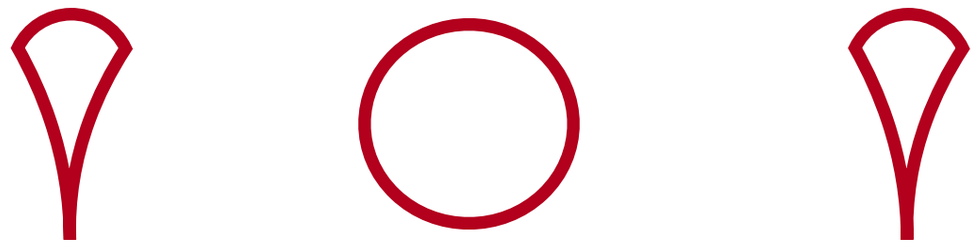
	Hierarchical Geometry	Measurement Geometry	Sparse Data
Centered Parameterization			

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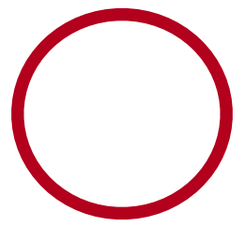
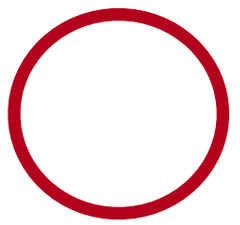
	Hierarchical Geometry	Measurement Geometry	Sparse Data	Dense Data
Centered Parameterization				

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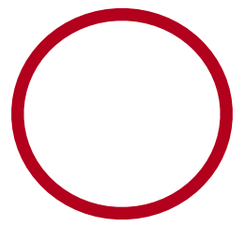
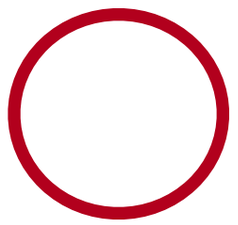
Centered Parameterization				
Non-Centered Parameterization				

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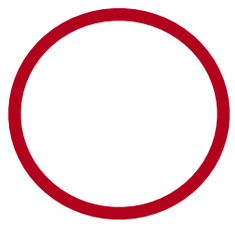
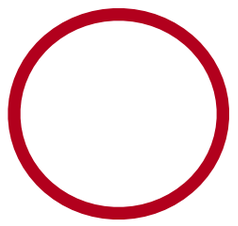
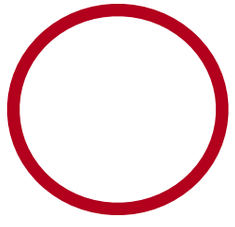
Centered Parameterization				
Non-Centered Parameterization				

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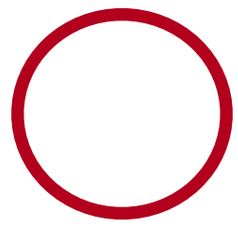
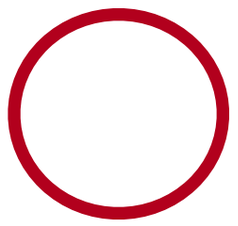
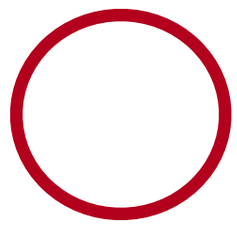
Centered Parameterization				
Non-Centered Parameterization				

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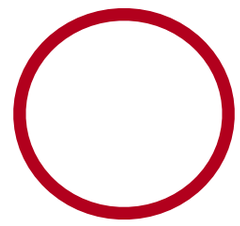
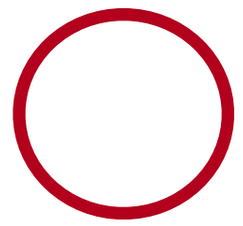
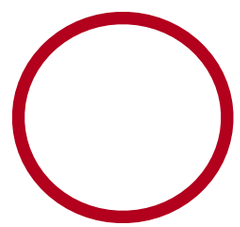
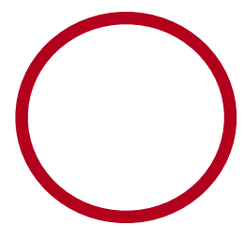
Centered Parameterization				
Non-Centered Parameterization				

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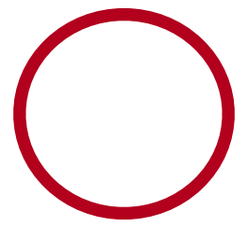
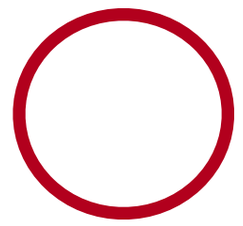
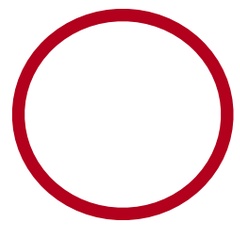
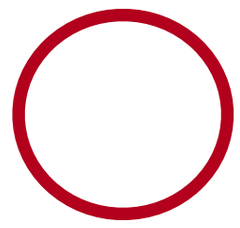
Centered Parameterization				
Non-Centered Parameterization				

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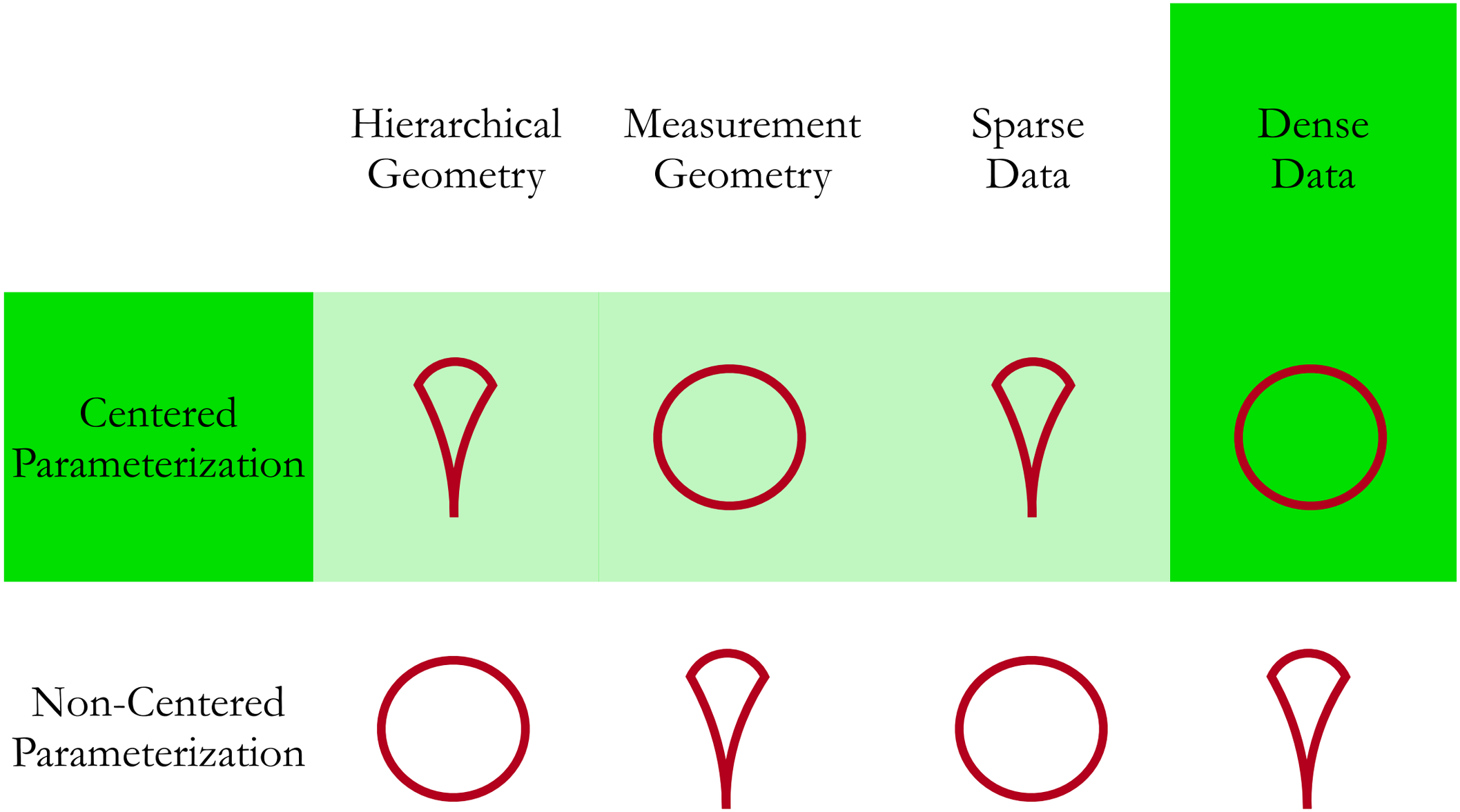

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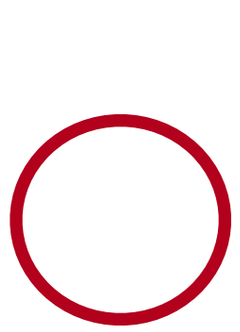
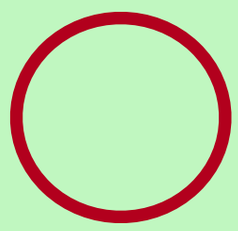
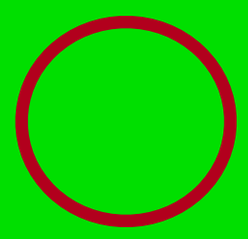
Centered Parameterization				
Non-Centered Parameterization				

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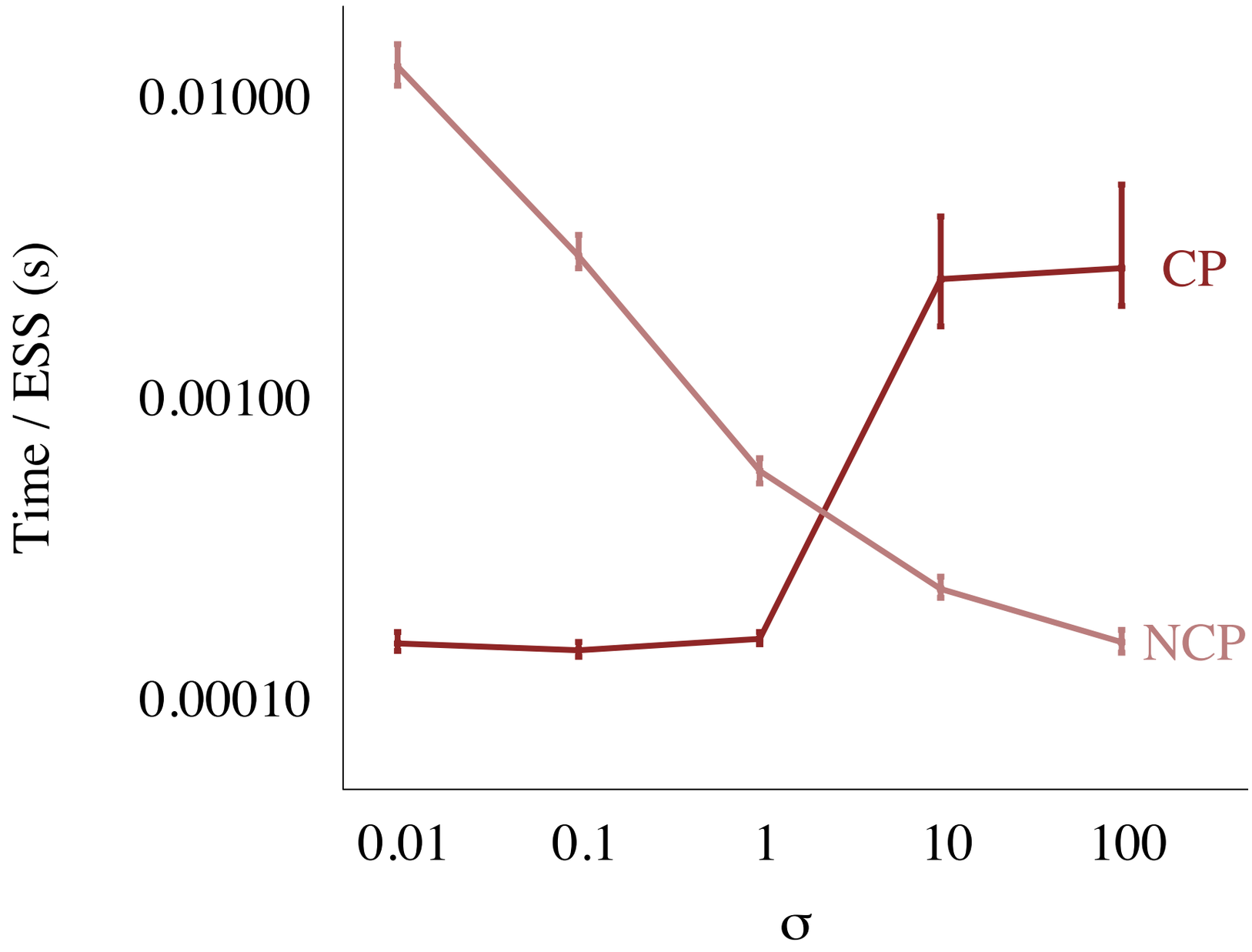
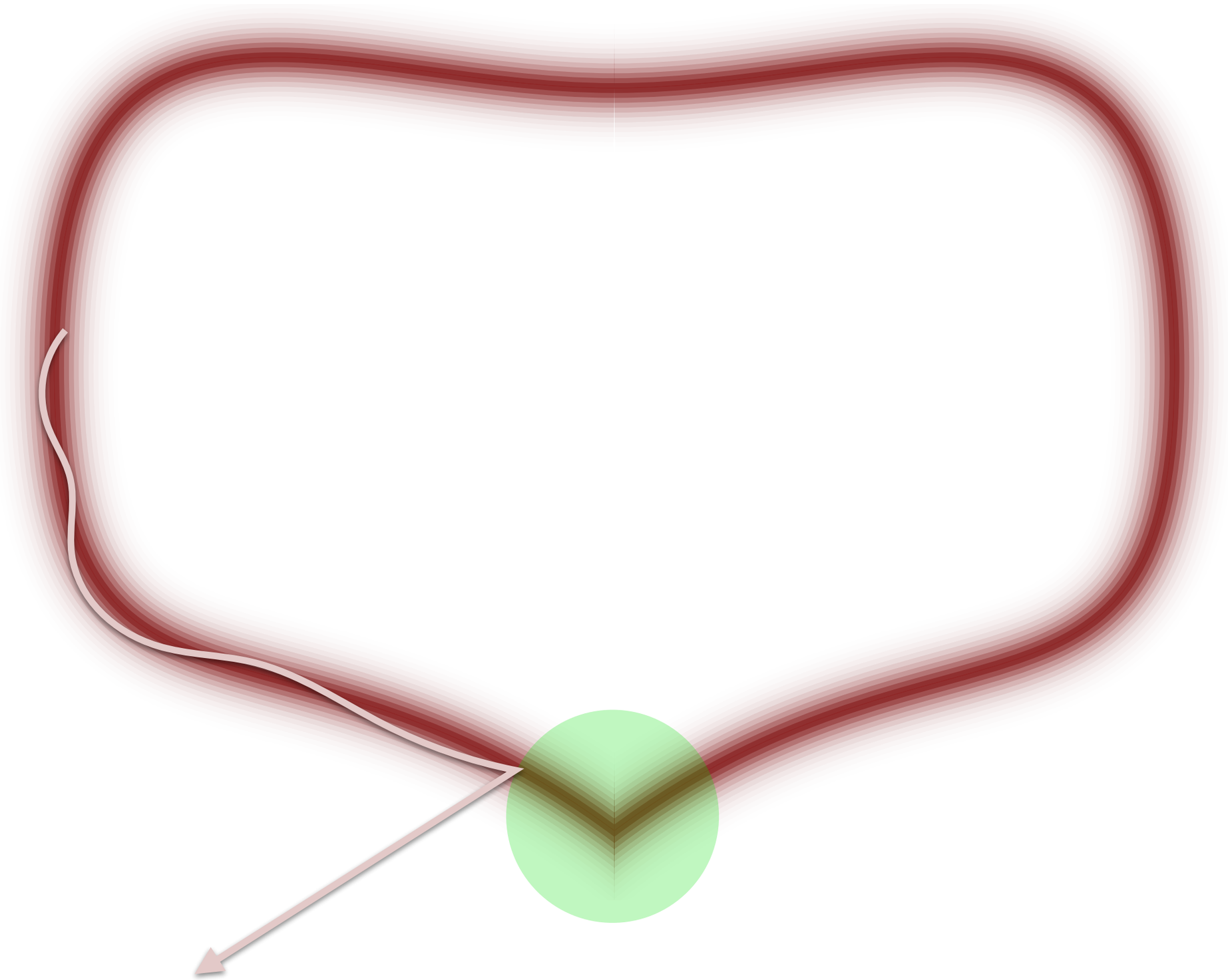
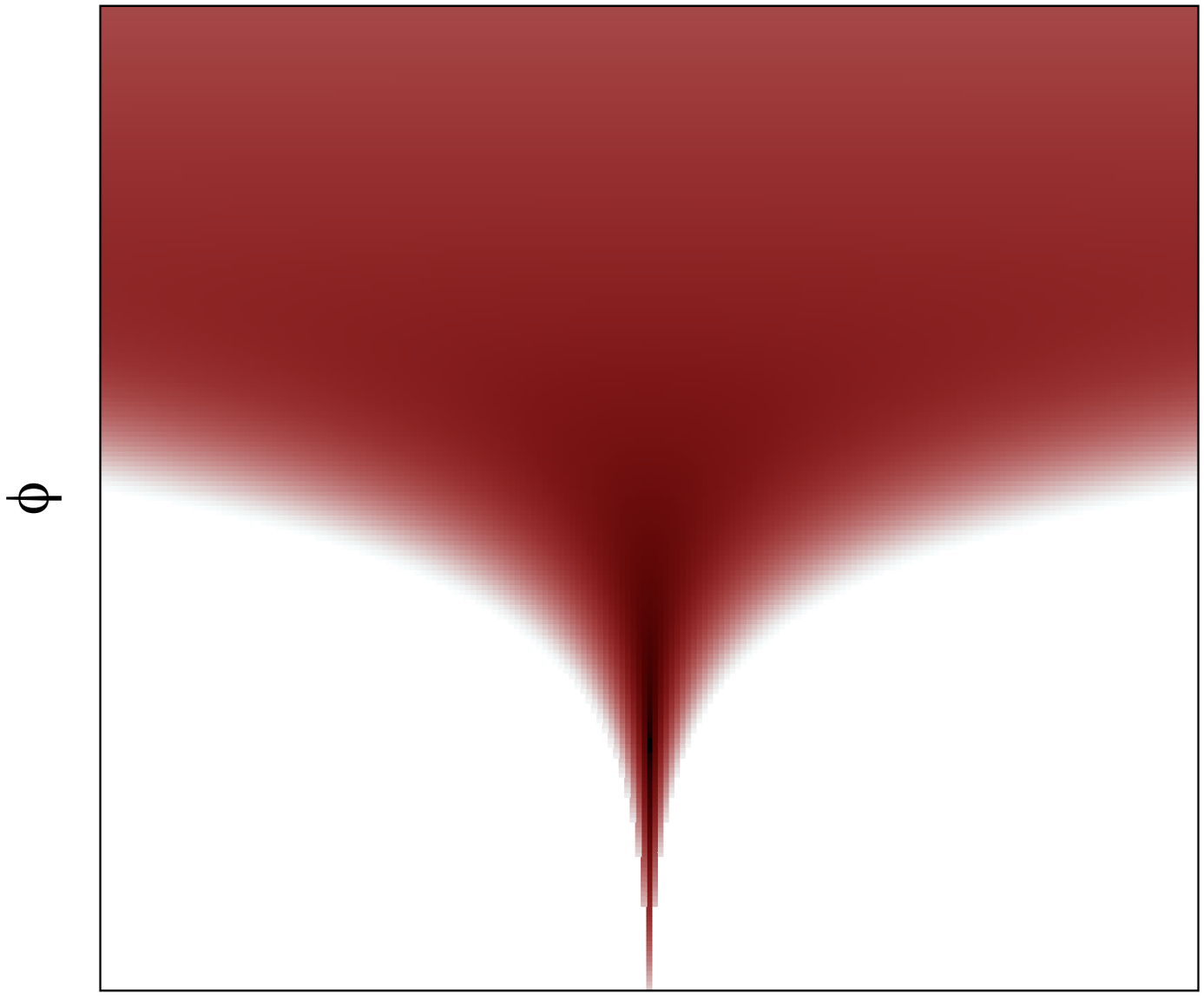


In practice, careful consideration of MCMC diagnostics help to identify poorly chosen parameterizations.





θ_n