Non integrable cases in satellite Dynamics

M. Eugenia Sansaturio
E.T.S.I.I., Paseo del Cauce s/n, E-47011 Valladolid, Spain
e-mail: marsan@wmatem.eis.uva.es

José M. Ferrándiz and Isabel Vigo–Aguiar
Dept. de Matemática Aplicada, Escuela Politécnica Superior
E-03080 Alicante, Spain. e-mail: josfer@altea.dma.ua.es

Abstract. In this paper, several questions related to the non integrability of some Hamiltonians appearing in the study of the problem of a satellite of a planet of arbitrary shape and recently developed by the authors are collected. Among them, we can quote the non integrability through meromorphic integrals of any truncation of the zonal satellite problem, as well as the non integrability through rational integrals of the J_{22}-problem.

1. Introduction

The search for the integrability of a given problem is a classical issue. Although non integrability may be expected as a generic property, its proof is usually hard to obtain even for simple cases and, in the last decades, a lot of mathematical tools have been developed towards this end (Ziglin, 1983; Yoshida, 1983-87-88-89). The three and n–body problems are those most profusely treated in the Celestial Mechanics literature. However, in recent years, some attention has been paid to the satellite problem, i.e., the orbital motion of a point mass around a planet of arbitrary shape.

In 1993, Irigoyen and Simó succeeded in proving the non integrability of the main problem of the satellite of an oblate primary, or J_2–problem, and this was a highly expected result in the scientific community. Basically, they applied a Theorem by Yoshida (1987), that was based on Ziglin's theorem (1983) for special Hamiltonian systems with a homogeneous potential, and concluded that there was no additional global meromorphic first integral besides the Hamiltonian itself.

More recently, the authors started to consider more general cases of the satellite problem. In this paper we report on the partial results obtained so far (Ferrándiz and Sansaturio, 1995; Ferrándiz *et al.*, 1996), that concern the J_{22}–problem, the general zonal problem and that with an arbitrary tesseral perturbation. Although these problems are not homogeneous either with respect to the properties required for the first integrals or with respect to the concrete approach and theorems applied in each case, the basic method is always Ziglin's theorem.

In section 2 we consider the problem of the zonal satellite. The aforementioned result about the non integrability of the J_2–problem is generalized to a perturbation made up of an arbitrary number of zonal harmonics. In this way, the non integrability through meromorphic integrals of the zonal satellite problem truncated at any order is shown. It is worthwhile noticing that when the number of harmonics is infinite, there could be complete integrability, as occurs in Vinti's problem (Vinti, 1960-69). Such a problem, closely related to Euler's problem of two–fixed centres of force (Szebehely, 1967), provides an example of a non trivial integrable Hamiltonian for which all the truncations of order $n > 2$ (of the development of the potential in spherical harmonics) are non–integrable in the Liouville sense. A similar case appears in the Toda lattice, as was shown by Yoshida (1988), and the non integrability of truncations is expected to be a rather generic property, although the number of examples is limited. The inverse problem looks more interesting since, up to now, the existence of other infinite series of the development in zonal harmonics giving rise to complete integrability, apart from those of Vinti or the trivial one (Kepler's problem), is not known. In this respect, the heavy rigid body problem seems to have given rise to more integrable subcases (Ziglin, 1980).

In section 3 we present a further elaboration of our former result on the J_{22}–problem, that is, the problem of a satellite only perturbed by the main sectorial harmonic, for which we establish the non–existence of additional global rational integrals essentially by applying a theorem due to Yoshida (1989) to an auxiliary problem. In spite of the fact that finding a body with the required moments of inertia and providing an example of such a problem is not a difficult task, this case can be mainly considered as a problem of mathematical interest. Nevertheless, it can also be considered as the first step in the treatment of those cases where the perturbation includes terms beyond the zonal ones and for which no component of the angular momentum is kept, so that no simplification is possible by means of a reduction in the number of degrees of freedom.

2. The truncated zonal satellite problem

Let us begin by recalling a non integrability criterion for 2–degrees of freedom Hamiltonian systems, with homogeneous potential of integer degree, obtained by Yoshida in 1987. Such a result is expressed as follows:

"*Theorem 1*: Let $V(q_1, q_2)$ be a homogeneous potential function of degree $k \in \mathbb{Z}$ and compute the quantity (integrability coefficient) λ defined by

$$\lambda = Tr\left(V_{\mathbf{qq}}(c_1, c_2)\right) - (k-1), \tag{1}$$

where $V_{\mathbf{qq}}$ is the Hessian matrix of $V\,(q_1, q_2)$ and $\mathbf{c} = (c_1, c_2)$ is a solution of the algebraic equation $\mathbf{grad}\, V(\mathbf{c}) = \mathbf{c}$.

If λ is in the so–called non integrability regions S_k, then the 2–degrees of freedom Hamitonian system $H = \frac{1}{2}\mathbf{p^2} + V\,(\mathbf{q})$ is non integrable, i.e. there cannot exist an additional integral Φ which is complex analytic in $(\mathbf{q}, \mathbf{p})$".

For our purposes, it suffices to consider $k \leq -3$ and the S_k given by

$$S_k = (1, \infty) \cup \left(\bigcup_{j \in \mathbf{N}} \left(\frac{-j\,(j+1)\,|k|}{2} + j + 1, \frac{-j\,(j-1)\,|k|}{2} - j + 1 \right) \right). \tag{2}$$

Now, let us consider the Hamiltonian of the truncated zonal satellite of order $n \geq 2$ in Cartesian canonical variables $(x,\ y,\ z;\ p_x,\ p_y,\ p_z)$

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - \frac{\mu}{r} + V(z, r(x, y, z))\,, \tag{3}$$

where

$$V = \sum_{k=2}^{n} \frac{\varepsilon_k}{r^{k+1}} P_k\left(\frac{z}{r}\right) \tag{4}$$

is the disturbing potential, $P_k(x)$ is the Legendre polynomial of order k, $r = \sqrt{x^2 + y^2 + z^2}$ and ε_k are coefficients which can be considered as small parameters.

If we carry out a change to cylindrical variables $(\rho,\ \phi,\ z;\ p_\rho,\ p_\phi,\ p_z)$ the Hamiltonian (3) is transformed into

$$H = \frac{1}{2}(p_\rho^2 + p_z^2) + \frac{p_\phi^2}{2\rho^2} - \frac{\mu}{r} + V(z, r(\rho, z))\,, \tag{5}$$

where $r = \sqrt{\rho^2 + z^2}$ and p_ϕ, the vertical component of the angular momentum, is a first integral since the coordinate ϕ is ignorable.

The Hamiltonian (5) has the integrals $H =$ const. and $p_\phi =$ const., which are independent and are in involution (i.e. the Poisson bracket $\{p_\phi, H\} = 0$). Let us suppose that there exists a third first integral F, independent of the other two and in involution with them, that is, verifying $\{p_\phi, F\} = 0$ since $\{F, H\} = 0$ always holds provided F is a first integral.

As $\{p_\phi, F\} = \partial F/\partial\phi$, it is obvious that F does not depend on ϕ. If we perform the reduction of order of the Hamiltonian (5), by considering $p_\phi = \alpha$ as a parameter, it is clear that if (5) is completely integrable with integrals H, p_ϕ and F, the reduction will also be so with integrals $H|_{p_\phi = \alpha}$ and $F|_{p_\phi = \alpha}$. Likewise, if the integrals of (5) are meromorphic, so are those of the reduction. Moreover, we can assume that F is analytical in $p_\phi = \alpha = 0$ since otherwise it suffices to multiply F by a suitable power of p_ϕ, as $p_\phi^k F$ is a first integral $\forall k$ because p_ϕ and F are so.

In consequence, if (3) is Liouville integrable, so is

$$H = \frac{1}{2}(p_\rho^2 + p_z^2) - \frac{\mu}{r} + V(z, r(\rho, z)) \,, \tag{6}$$

which is obtained by making $\alpha = 0$ in the reduction of (5) to two degrees of freedom.

Notice that, as the potential of this 2–degrees of freedom Hamiltonian consists of a finite number of homogeneous terms, whose degrees vary from -1 (corresponding to the Keplerian term) to $-n-1$, we are in conditions to apply the Yoshida theorem (1988, theorem 4.1), which allows us to establish the non–existence of an additional meromorphic integral if the integrability coeficient λ of either the lowest or highest order part is in their corresponding non integrability regions. To this end, proceeding as Yoshida (1983b), by performing a suitable change of scale, the Hamiltonian (6) becomes

$$K = \frac{1}{2}(p_\rho^2 + p_z^2) + V_n \,, \qquad \text{with} \qquad V_n = \frac{\varepsilon_n}{r^{n+1}} P_n\left(\frac{z}{r}\right) \,, \tag{7}$$

which is taken as Hamiltonian of an auxiliary problem (see Ferrándiz *et al.*, 1996, for details).

As the potential V_n is homogeneous of order $m = -n-1$, according to *Theorem 1*, the non-existence of any other meromorphic integral simply depends on finding a solution $\mathbf{c}$ of the algebraic equation

$$\mathbf{grad}\ V_n(\mathbf{c}) = \left(\frac{\partial V_n}{\partial \rho}(\mathbf{c}), \frac{\partial V_n}{\partial z}(\mathbf{c})\right) = \mathbf{c} \tag{8}$$

and on the value of the integrability coefficient λ defined by

$$\lambda = Tr\,(Hess\ V_n(\mathbf{c})) - (m-1) = n + 2 + Tr\,(Hess\ V_n(\mathbf{c}))\ . \tag{9}$$

It is easily checked that the system (8) admits a solution of the form $\mathbf{c} = (\phi = 0, z = z_0)$, where z_0 is a solution to the equation

$$\frac{-\,(n+1)}{\left(\sqrt{z_0^2}\right)^{n+3}} P_n\left(\frac{z_0}{\sqrt{z_0^2}}\right) = \frac{1}{\varepsilon_n}\,. \tag{10}$$

On the other hand, by using well known properties of Legengre polynomials, straightforward calculations allow us to compute the trace $Tr\,(Hess\ V_n(\mathbf{c}))$ to find that the integrability coefficient λ is

$$\lambda = n + 2 + Tr\,(Hess\ V_n(\mathbf{c})) = 1 + \frac{n}{2} > 1 \quad \forall n\,.$$

As the non integrability regions S_{-n-1} defined in (2) contain the interval $(1, \infty)$, we conclude that the auxiliary Hamiltonian (7) is non–integrable.

Now, coming back to the Hamiltonian (6), according to Yoshida (1988, theorem 4.1), in our case, it holds for the lowest order $-n-1$ and hence (6) is non–integrable. In consequence, as explained before, the original problem (5) is not Liouville integrable through meromorphic integrals.

Let us remark that the choice of the solution to (8) carried out here has allowed us to prove the non integrability of any truncation of the zonal satellite problem irrespective of whether it ends in even or in odd harmonics, while the solution chosen by Irigoyen and Simó (1993) to set up the non integrability of the J_2-problem (whose truncation ends in J_2) would only be useful to prove the non integrability of truncations ending in even harmonics.

3. Non integrability of the J_{22}–problem

The *Theorem 1* by Yoshida presented in the previous section is useful to establish the non integrability of Hamiltonian systems with homogeneous potential provided it has only two degrees of freedom. If it is not the case, other alternative procedures have to be used. In this line, Yoshida (1989) obtained a criterion, also based on Ziglin's theorem (Ziglin, 1983), for the non–existence of an additional meromorphic integral in Hamiltonian systems with n degrees of freedom. Such a criterion can be stated as follows:
"*Theorem 2*: Consider a Hamiltonian system with n degrees of freedom of the form

$$H = \frac{1}{2}\mathbf{p}^2 + V(\mathbf{q})\,, \tag{11}$$

where $V(\mathbf{q})$ is assumed to be a homogeneous function of integer degree k ($k \neq 0, \pm 2$). Since the equations of motion are scale–invariant, one can compute the Kowalevski exponents (KE), $\rho_1, \rho_2, \cdots, \rho_{2n}$ (see Yoshida, 1983a). The KE characterize the branching of the solution in the complex t–plane as follows,

$$\mathbf{q}(t) = t^{-g}\left[\mathbf{d} + \mathrm{Taylor}(I_1 t^{\rho_1}, I_2 t^{\rho_2}, \cdots)\right]\,. \tag{12}$$

Here, $g = 2/(k-2)$, $\mathbf{d}$ is a constant vector, I_j are integration constants, and the term $\mathrm{Taylor}(x, y, z, \cdots)$ denotes a convergent Taylor series. For (11), the KE have a pairing property, $\rho_i + \rho_{i+n} = 2g+1$ ($i = 1, 2, \cdots, n$). Because of this pairing property, ρ_i and ρ_{i+n} can be rational numbers if and only if the difference $\Delta\rho_i := \rho_{i+n} - \rho_i$ is a rational number. Further, one can always assume that $\Delta\rho_n$ is rational, $\Delta\rho_n = (3k-2)/(k-2)$.

Then, if the numbers $(\Delta\rho_1, \Delta\rho_2, \cdots, \Delta\rho_n)$ are rationally independent, then the Hamiltonian system (11) has no additional global analytic integral $\Phi(\mathbf{q}, \mathbf{p}) = const$ besides the Hamiltonian itself".

In general, it is quite obvious that the greater the number of degrees of freedom is, the more complicated the problem becomes. Moreover, let us point out that the analysis carried out through the above theorem is much more involved than that performed in the previous section, in which we only had to compute the integrability coefficient, while in the n–degrees of freedom criterion the knowledge of the eigenvalues of an n–dimensional matrix is required to determine the rational independence of the KE.

Next, we will consider the case for the satellite problem in which the only acting perturbation is that due to the term corresponding to the harmonic (2,2) of the development in spherical harmonics of the potential of the planet (i.e. the J_{22}–problem).

By using Cartesian coordinates in the rotating system $Oxyz$, attached to the rigid body, and which turns with uniform angular velocity ω with respect to the inertial axis, the Hamiltonian of the problem is

$$H_s = \frac{1}{2}|\mathbf{p}|^2 - \frac{\mu}{r} + V_{22} - \omega(xp_y - yp_x)\,, \tag{13}$$

where

$$V_{22} = \frac{\varepsilon(x^2 - y^2)}{r^5}\,, \quad |\mathbf{p}|^2 = p_x^2 + p_y^2 + p_z^2\,, \quad r = |\mathbf{q}| = \sqrt{x^2 + y^2 + z^2}$$

and the term $\omega(xp_y - yp_x)$ is due to the rotation of the reference frame.

Since the available non integrability results are not directly applicable to the original problem (13), we first consider an auxiliary problem obtained by the reduction of the system deduced from (13) to a similarity invariant one. This system will be proved not to have additional meromorphic integrals by using the above quoted *Theorem 2*.

By performing the change of scale $(t, \mathbf{q}, \mathbf{p}) \longrightarrow (\overline{t}, \overline{\mathbf{q}}, \overline{\mathbf{p}})$ defined by $t = \alpha^{-1}\overline{t}$, $\mathbf{q} = \alpha^{-2/5}\overline{\mathbf{q}}$, $\mathbf{p} = \alpha^{3/5}\overline{\mathbf{p}}$ and taking limits when $\alpha \longrightarrow \infty$, the Hamiltonian (13) becomes

$$K_s = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \varepsilon\frac{(x^2 - y^2)}{r^5}\,, \tag{14}$$

which is taken as Hamiltonian of an auxiliary problem.

Taking into account that (14) is of the type (11), with homogeneity degree $k = -3$, we can apply the *Theorem 2*. Since the algebraic equation **grad** $V(\mathbf{c}) = \mathbf{c}$ admits as solutions $\mathbf{c} = (0, \sqrt[5]{3\varepsilon}, 0)$ and $\mathbf{c} = (\sqrt[5]{-3\varepsilon}, 0, 0)$, we can compute the Hessian $V_{\mathbf{qq}}(\mathbf{c})$ and its eigenvalues to find that, in this case, $\Delta\rho_1 = \mathrm{i}\sqrt{31}/5$, $\Delta\rho_2 = \mathrm{i}\sqrt{15}/5$ and $\Delta\rho_3 = 11/3$. Therefore, as the imaginary numbers $\mathrm{i}\sqrt{31}$, $\mathrm{i}\sqrt{15}$ are rationally independent, the auxiliary Hamiltonian K_s (14) does not have any global meromorphic integral independent of the Hamiltonian.

Now, starting from the non integrability of the auxiliary problem, we will try to get as much information as possible about the non integrability of the original Hamiltonian (13). Let us remember that the forementioned theorem by Yoshida (1989) is proved by using a particular straight–line solution of the form $\mathbf{q}(t) = \mathbf{c} \cdot \phi(t)$, with $\phi(t)$ satisfying $\phi'' + \phi^{k-1} = 0$. Afterwards, the linear variational equations around that solution are set up. According to Ziglin's result (Ziglin, 1983), the problem is not integrable if two different non resonant monodromy matrices (corresponding to suitable loops on a certain Riemann surface) not commuting can be found. In the case of homogeneous potential, the normal variational equations can be transformed into a Gauss hypergeometric equation and explicit expressions for the monodromy matrices are known (Yoshida, 1989).

However, in our problem, the presence of the gyroscopic term in ω makes the aforementioned analysis more involved. In fact, it is not trivial to find a particular solution which, in the limit $\omega \longrightarrow 0$, tends to the straight–line solution used by Yoshida, which is decisive for calculating the monodromy matrices explicitly - apart from possible inconvenient variations of the Riemann surface and the necessary loops as ω varies. So, alternative procedures should be used to ensure that (13) is not analytically integrable.

Nevertheless, what we can easily conclude is the non–existence of additional rational integrals in the original problem (13). Let us first note that if $\Phi(\mathbf{p}, \mathbf{q}, t) = const.$ is a rational first integral, by performing the forementioned change of variables, it becomes

$$\overline{\Phi}(\overline{\mathbf{p}}, \overline{\mathbf{q}}, \overline{t}) = \frac{\displaystyle\sum_m \alpha^m f_m}{\displaystyle\sum_n \alpha^n f_n} = const., \tag{15}$$

where f_m and f_n are polynomials in $\overline{\mathbf{p}}$, $\overline{\mathbf{q}}$, t.

The expression (15) is a rational first integral of (13). Then, multiplying by an adequate power of α and taking limits when $\alpha \longrightarrow \infty$, we obtain a rational integral of the auxiliary problem (14).

Since (14) does not have any additional meromorphic integral, it cannot admit any additional rational integral and thus, neither does (13). Notice that the same assertion still holds if $\mathbf{q}$ gathers not only the Cartesian coordinates but also the radius r (or any other homogeneous function in the coordinates).

Another result, not formulated before, is the non–existence of any additional meromorphic integral without poles in $z = 0$ and $\omega = 0$ since, otherwise, the planar J_{22}–problem would have a meromorphic integral which is analytical in $\omega = 0$. But this is impossible and can be easily deduced from Yoshida (1988, theorem 4.1). Moreover, we have the feeling that it could be quite difficult to have a spatial J_{22}–problem without the restriction of non–singularity in $z = 0$ and $\omega = 0$.

Let us remark that the case under study in this section is an example of sectorial truncation of the satellite problem (i.e. the problem of the satellite only perturbed by the main tesseral harmonic J_{22}). In this respect, new results concerning the non integrability of sectorial and tesseral truncations of the satellite problem, when more suitable coordinates than the Cartesian ones are used, have been recently obtained by the authors and will be published in a forthcoming paper (Sansaturio *et al.*, 1996).

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