NON–INTEGRABILITY OF THE MOTION OF A POINT MASS AROUND A PLANET OF ARBITRARY SHAPE

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Abstract. In this paper, several results on the non–integrability of the problem of a satellite under different assumptions on the mass distribution of the primary are collected. Among them, we pay special attention to the non–integrability through meromorphic integrals of any truncation of the zonal satellite problem, as well as to that of the $(J_2 + J_{22})$–problem or 'general main satellite problem'.

1. Introduction

More than one century ago, the theory of orbits went through an important qualitative change when Bruns and then Poincaré (see [10]) proved by means of several methods and hypotheses that the three body problem has no more first integrals apart from the ten classical ones. Since then, great progress has been made in the knowledge of the behaviour of such non–integrable systems (including chaos), as well as in the methods which allow us to investigate the non–integrability. A panorama of the evolution of the ideas and results concerning the study of problems of several bodies can be found in the usual treatises – for instance, see [9] or [5].

However, questions about non–integrability of problems of satellites of external bodies remained open till this decade. Irigoyen and Simó [3] published the first result along this line, proving that the main problem of the satellite of an oblate primary, or J_2–problem, was not completely integrable through meromorphic integrals. The proof was based on some theorems by Ziglin [14] and Yoshida [11], which will be commented on briefly in section 2. Later, an analogous method allowed Ferrándiz and Sansaturio [1] to establish that the J_{22}–problem did not have a complete system of first integrals. Such integrals were supposed to be rational since, in three dimensions, the non–integrability according to Ziglin depends on a condition of rational independence of the eigenvalues of certain matrices which does not allow us to use the continuity arguments as happens in dimension 2.

In [2], the present authors set up the non–integrability through meromorphic integrals of the truncation of the zonal satellite problem at any order. Then in [6], we solved the case in which the perturbation was an arbitrary sectorial harmonic and established that the main satellite problem, or $(J_2 + J_{22})$–problem, is not completely integrable either.

In this paper we gather together the above mentioned results, either with much briefer proofs or by generalizing them.

2. Brief Summary of Yoshida's Theorem

Very broadly speaking, Ziglin's method [14] takes into consideration an analytic Hamiltonian for which a particular solution describing a curve non–homologous to zero on a certain Riemann surface is selected. Then, the non–integrability depends on the properties of the monodromy group associated to the variational equations along such a solution, and particularly on certain non–resonance conditions.

In the case of a Hamiltonian

$$H = \frac{1}{2}\mathbf{p}^2 + V(\mathbf{q}) \,,$$

where $V(\mathbf{q})$ is homogeneous of an integer degree $k \neq 0, \pm 2$, Yoshida [11] obtained very useful criteria, based on Ziglin's results, by making use of the existence of a straight–line solution of the form

$$\mathbf{q} = \mathbf{c}\phi(t) \,, \qquad \mathbf{p} = \mathbf{c}\dot{\phi}(t) \,,$$

where $\phi(t)$ is a solution of $\ddot{\phi}(t) + \phi^{k-1} = 0$ and the constant vector $\mathbf{c}$ is a solution to the algebraic equation $\mathbf{c} = V_{\mathbf{q}}(\mathbf{c})$.

In a standard way, the variational equations can be transformed into a Gauss hypergeometric equation, with well known monodromy group and finally, the non–integrability depends on the eigenvalues of $V_{\mathbf{qq}}(\mathbf{c})$, one of them always being equal to $k - 1$.

In the planar case, $n = 2$, the problem turns out to be non–integrable if a coefficient referred to as the integrability coefficient, $\lambda = \mathrm{Tr}V_{\mathbf{qq}}(\mathbf{c}) - (k-1)$, lies in certain non–integrability regions S_k ([11]; p. 128, eq. (1.3)), which are derived from the fact that the trace of some monodromy matrices is > 2. For $n > 2$, as shown in [13], non–integrability follows if the parameters

$$\Delta\rho_i = \sqrt{1 + \frac{8k\lambda_i}{(k-2)^2}}$$

are rationally independent, where the λ_i are the eigenvalues of $V_{\mathbf{qq}}(\mathbf{c})$.

The reformulation of these criteria into polar or spherical coordinates presented in [6] has turned out to be useful in dealing with orbital problems, since it allows us to simplify the calculations, sometimes very drastically. In the planar case, the integrability coefficient in polar coordinates is easily computed through

$$\lambda_p = \frac{W_{\theta\theta}(\theta_0)}{kW(\theta_0)},$$

where $V(r,\theta) = r^k W(\theta)$ is the potential function, k being an integer but $k \neq 0, \pm 2$, and θ_0 is a solution to $W_\theta(\theta) = 0$ such that $W(\theta_0) \neq 0$. This coefficient relates to Yoshida's integrability coefficient through the equation $\lambda_p = \lambda - 1$.

3. Non–integrability of the Truncated Zonal Satellite Problem

In this section we present some results about this problem recently obtained by the authors [2], although the proof has been considerably shortened.

By using cylindrical coordinates $(\rho, \varphi, z; p_\rho, p_\varphi, p_z)$, the homogeneous Hamiltonian for this problem can be written as

$$H = \frac{1}{2}\left(p_\rho^2 + p_z^2\right) + \frac{p_\varphi^2}{2\rho^2} - \frac{\mu}{r} + \sum_{k=2}^{n} \frac{V_k}{r^{k+1}} + p_0\,, \qquad (1)$$

where $r = \sqrt{\rho^2 + z^2}\,$, $V_k = \varepsilon_k P_k(z/r)\,$, ε_k are constant and $P_k(x)$ is the Legendre polynomial of order k.

From the fact that p_φ is a first integral of (1) it follows that, if this Hamiltonian admits a meromorphic integral independent from the other two and in involution with them, such an integral is analytic with respect to p_φ. Consequently, the integrability in the Liouville sense of (1) implies that of the Hamiltonian

$$H = \frac{1}{2}\left(p_\rho^2 + p_z^2\right) - \frac{\mu}{r} + \sum_{k=2}^{n} \frac{V_k}{r^{k+1}} + p_0\,, \qquad (2)$$

which is obtained by making $p_\varphi = 0$ in (1). It is immediate that (2) admits a particular solution with $\rho = p_\rho = 0$. However, since an explicit expression for the associate monodromy matrices which are required by Ziglin's method is not known, it is convenient to introduce a suitable limit problem.

The change of scale defined by

$$t = \beta \bar{t}\,, \qquad \rho = \beta^{\frac{2}{n+3}}\,\bar{\rho}\,, \qquad z = \beta^{\frac{2}{n+3}}\,\bar{z}\,,$$
$$p_0 = \beta^{-\frac{2(n+1)}{n+3}}\,\bar{p}_0\,, \qquad p_\rho = \beta^{-\frac{n+1}{n+3}}\,\bar{p}_\rho\,, \qquad p_z = \beta^{-\frac{n+1}{n+3}}\,\bar{p}_z\,,$$

gives a new homogeneous Hamiltonian, which after multiplying by $\beta^{\frac{2(n+1)}{n+3}}$ and taking limits as $\beta \to 0$, reduces to

$$K = \frac{1}{2}\left(p_\rho^2 + p_z^2\right) + V + p_0\,, \quad \text{where} \quad V = \frac{\varepsilon_n}{r^{n+1}} P_n\left(\frac{z}{r}\right) \tag{3}$$

and the bar in the new variables has been removed to simplify the notation.

The Hamiltonian (3) is of the form required by Yoshida's theorem [11], since the potential V is homogeneous of degree $k = -n-1$. As is well known, the Hessian of V computed on the vector $\mathbf{c}$ quoted in section 2 always has the eigenvalue $k-1$, which in our case turns out to be $-n-2$. It is easy to establish that this is precisely the value of $\partial^2 V/\partial z^2$, so that the integrability coefficient reduces to $\partial^2 V/\partial \rho^2$. Moreover, due to the symmetry with respect to the OZ axis, it is obvious that

$$\frac{\partial^2 V}{\partial \rho^2} = \frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial y^2} \quad \text{in} \quad \rho = x = y = 0\,,\ z = c\,.$$

Since V verifies Laplace equation, we get

$$\left.\frac{\partial^2 V}{\partial \rho^2}\right|_{\substack{\rho = 0 \\ z = c}} = -\frac{1}{2}\left.\frac{\partial^2 V}{\partial z^2}\right|_{\substack{\rho = 0 \\ z = c}} = 1 + \frac{n}{2}\,,$$

a value which lies in the non–integrability region S_{-n-1} and coincides with that previously obtained by the authors in [2] but with rather more tedious calculations.

Once we have established that the limit problem (3) does not have a complete system of meromorphic integrals, it follows that neither does the problem (2) in a neighbourhood of $\beta = 0$.

Now, it is enough to apply the usual continuity argument – see [12] or [3] –, since the non–integrability regions correspond to those for which the trace of the monodromy matrices is > 2.

However, notice that such an argument is more complicated than simply checking the aforementioned inequality. In fact, it is necessary to verify that the Riemann surface, the poles and the loops which are used to calculate the monodromy matrices do not suffer significant qualitative changes for any problem close to the limit problem, i.e., in a neighbourhood of $\beta = 0$

after performing the change of scale. Nevertheless, this can be verified in an immediate way.

To this end it is enough to realize that the basic solution can be considered of the form $\rho = 0$, $z = \psi(t)$, where ψ is a solution of the differential equation

$$\ddot{\psi} + \frac{\partial V}{\partial z}(0, \psi) = 0\,,$$

which admits as a first integral

$$\frac{1}{2}(\dot{\psi})^2 + V(0, \psi) = h\,,$$

and that the Riemann surface involved in Ziglin's theorem is defined by the integral

$$t = \int \frac{dw}{\sqrt{p(w)}}\,, \quad \text{with } p(w) = 2(h - V(0, w))\,.$$

The monodromy group of the normal variation equation is then built by taking closed circuits around the branch points corresponding to the roots of $p(w) = 0$.

In the limit problem, we can take $h = \varepsilon_n\, P_n(\psi/\sqrt{\psi^2})$ and $n+1$ simple poles, roots of $w^{n+1} = 1$, appear. If the potential is a finite sum of zonal harmonics it is obvious that the solution $\rho = 0$ still exists and the equation $w^{n+1} = 1$ would be replaced by another one with the same leading and independent terms and with lower powers of w multiplied by factors which are positive powers of $\beta^{\frac{2}{n+3}}$ and hence it goes on having $n+1$ simple roots provided β is small enough.

Notice that this result includes as a particular case the non–integrability through meromorphic integrals of the truncated two fixed centres problem in the asymmetric case. The non–integrability of such a problem in the symmetric case was first established by Irigoyen [4] by following a scheme similar to that used for the J_2–problem in [3]. We would like to point out that the starting solution used in Irigoyen's work is not suitable for the study of odd harmonics.

On the other hand, although in (1) only the harmonics with negative powers have been included, it is easy to realize that if we add up a finite number of zonal harmonics of the form $r^k V_k$ to (1), the conclusion still holds. Such terms could be envisaged as coming from the truncation of the development of a perturbation due to a third body by simply assuming some suitable conditions (e.g. the Keplerian orbit is circular and equatorial).

4. The General Main Satellite Problem

In this section we slightly generalize some results previously obtained by the authors [6] in the sense that we succeed in proving that the (J_2+J_{22})–problem is not completely integrable through meromorphic integrals.

Assuming that the primary rotates with constant angular velocity ω around the Oz axis and using Cartesian coordinates in the rotating system $Oxyz$ attached to the primary, the Hamiltonian is of the form

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - \omega(xp_y - yp_x) - \frac{\mu}{r} + V_2 + V_{22}\,, \tag{4}$$

with

$$V_2 = \frac{\varepsilon_2}{r^3} P_2\left(\frac{z}{r}\right)\,, \qquad V_{22} = -\frac{\varepsilon_{22}}{r^5}(x^2 - y^2)$$

and ε_2, ε_{22} are constants which are proportional to the small parameters J_2, J_{22}, respectively. Notice that whenever the moments of inertia verify $A < B < C$, as in the case of the Earth, ε_2 and ε_{22} are positive.

Since any solution with initial conditions $z = p_z = 0$ always remains in the plane $z = 0$, by making $z = p_z = 0$ in (4), we obtain a new Hamiltonian H_0 whose flow is included in that of (4).

In polar coordinates such a Hamiltonian H_0 is expressed as

$$H_0 = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \omega p_\theta - \frac{\mu}{r} - \frac{\varepsilon_2}{2r^3} - \frac{\varepsilon_{22}}{r^3}\cos 2\theta\,. \tag{5}$$

It is immediate that if (4) has a first integral which is analytic in $z = 0$, $p_z = 0$ and is independent from H, it will give rise to another first integral which is independent from H_0 for the subproblem (5). On the other hand, if this integral was analytic in ω, at least in $\omega = 0$, it would also provide an integral of the auxiliary problem

$$K_0 = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{\mu}{r} - \frac{\varepsilon_2}{2r^3} - \frac{\varepsilon_{22}}{r^3}\cos 2\theta\,, \tag{6}$$

corresponding to a non–rotating primary.

The problem (6) can be treated in a very similar way to that of the zonal harmonics. A suitable change of scale allows us to go to a limit problem, in which the Keplerian term disappears, that is

$$K_1 = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \frac{1}{r^3}W(\theta)\,, \quad \text{with} \quad W(\theta) = -\frac{\varepsilon_2}{2} - \varepsilon_{22}\cos 2\theta\,. \tag{7}$$

The integrability coefficient in polar coordinates is extremely easy to calculate. According to what is indicated in section 2, it is enough to consider the solution $\theta_0 = 0$ to $W_\theta(\theta) = 0$. Therefore,

$$\lambda_p = \frac{W_{\theta\theta}(0)}{(-3)W(0)} = \frac{4\varepsilon_{22}}{3(\varepsilon_2/2 + \varepsilon_{22})} > 0\,.$$

This coefficient lies in one of the non–integrability regions given in [6], so that (7) does not have any additional meromorphic integral.

The usual continuity argument proves that (6) also verifies this property. This allows us to conclude that the original problem (4) does not have any global first integral which is meromorphic and also analytic in $w = 0$, and in $z = p_z = 0$. This result generalizes those previously obtained by the authors ([2], [6]), where only the non–existence of additional rational integrals was proved, either for the case $\varepsilon_2 = 0$ or for the general case.

5. Final comments

There are several questions of practical interest which could arise in connection with the non–integrability, such as the size of the chaotic zones (as it was established by Simó [8] in the J_2–problem) and, in general, its relevance from a practical point of view. Most of these questions are still open and the answers do not seem to be trivial. Some considerations about the physical meaning of the orbits required by Ziglin's method, as well as some numerical simulations can be seen in [7], where it is shown the appearance of chaos (hard or soft) for moderate values of J_{22} in the neighbourhood of the 1:1 resonant orbit.

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