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A DIRECT METHOD TO CALCULATE THE ENERGY EVOLUTION OF A TURBULENT FLOW

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Abstract

The classical configuration of the power spectrum of a homogeneous turbulent flow provides a functional relation between energy and enstrophy which may be used to find the evolution of energy in the intermediate stages of finite time and viscosity to which the usual asymptotic arguments do not apply. The method has a minimal computational cost compared with the direct numerical integration of the whole Navier-Stokes system. It is applied here to study the energy evolution both theoretically and numerically, obtaining the minima of energy compatible with the power spectra as well as the rate of decay of the energy towards them. The results may be useful to study the compatibility of the Kolmogorov and Kraichnan spectra with the observed energy evolution of a turbulent flow.

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1 Introduction

While the spatial behavior of turbulent fluids is by definition extremely complicated, there are at least two features which are actually more predictable for turbulent than nonturbulent flows. One of them is the power spectrum, and the other the time asymptotics in unforced homogeneous turbulence. Let us first recall briefly the main facts of the problem: unforced incompressible fluids satisfy the Navier-Stokes equations, written after normalization of constants as

$$\frac{\partial \mathbf{u}}{\partial t} = \nu \nabla^2 \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u} - \nabla p$$
$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

where $\mathbf{u}$ is the fluid velocity and p the pressure. For any domain U with Dirichlet boundary conditions $\mathbf{u} \mid_{\partial U} = 0$, or for the whole space with adequate hypotheses of decrease at infinity, the kinetic energy

$$E = \frac{1}{2} \int_U |\mathbf{u}|^2 \, dV \tag{2}$$

satisfies the equation

$$\frac{dE}{dt} = -\nu \Omega, \tag{3}$$

where

$$\Omega = \int_U |\nabla \mathbf{u}|^2 \, dV \tag{4}$$

is the fluid enstrophy. When the domain U is bounded, the energy decays at least exponentially in time. This goes as well for periodic problems: however, for the whole space, which we will study from now on, the problem is much more complicated. Batchelor and Proudman [1] argued that the energy of turbulent flows of zero mean decays like $t^{-5/2}$, while Saffman [2] proposed $t^{-3/2}$ for flows of nonzero mean. For two-dimensional turbulence the scaling should behave respectively as t^{-2}, t^{-1}. Although their arguments are partly heuristic, it is worth noting that essentially the same results have been proved rigorously for any flow [3]. From the viewpoint of turbulence, however, this limit is not very interesting. In the last stages of its evolution the flow has in fact stopped to be turbulent and is merely a laminar one with complex structure. Instead the asymptotics studied are as follows: take a fixed time large enough for turbulence to have developed, and consider the limit of the energy as a function of the viscosity ν when ν tends to zero. Since viscosity is usually rather low for turbulence to exist, this theoretical limit is the one really found in experiments and modelizations for much of the fluid evolution. To find its precise value we need to address a second important aspect of turbulent flows: the power spectrum. If we define $E(k)$ as the energy of all the modes with frequency of modulus k,

$$E(k) = \int_{|\mathbf{k}|=k} |\hat{\mathbf{u}}(\mathbf{k})|^2 \, d\sigma(\mathbf{k}), \tag{5}$$

where $\hat{\mathbf{u}}$ denotes the Fourier transform of $\mathbf{u}$, the function $k \to E(k)$ is called the power or energy spectrum of the flow; naturally it depends on time. The invariance of the unforced Navier-Stokes equation under the scaling $\mathbf{x} \to l\mathbf{x}$, $t \to l^{1-h}t$, $\mathbf{u} \to l^h\mathbf{u}$, $\nu \to l^{1+h}\nu$ yields the following result [4]: the energy must have the form

$$E(k, \nu) = k^q g(kt^a, \nu t^b),$$

for a certain function g, where $a = 2/(3 + q)$, $b = -(1 + q)/(3 + q)$. For very small viscosity, the dependence of g on its second variable could be ignored [5]. Obviously the value of q cannot be the same for different parts of the spectrum: for the inertial range it should agree with the Kolmogorov value $q = -5/3$, whereas such negative exponent would yield infinite energy if it holds in the integral (subinertial) range. Thus q should be found in every portion of the spectrum by modelization and observation, as in [5]. For infinite domains q is expected to be 2, and the next power in the Taylor expansion of E should be 4:

$$E(k, t) = B_0k^2 + B_2(t)k^4 + o(k^4). \tag{6}$$

This expansion should hold in dimension three, while the powers of k decrease by one in dimension two: this is because the area of the sphere of radius k scales like k^2, whereas the circumference length does with k. The reason to concentrate in these powers is that the coefficients B_0 and B_2 have a precise physical interpretation and can be expected to be robust, which is not so clear for $q < 2$ [6]. B_0 is the mean of $\mathbf{u}$, which is constant by linear momentum conservation; whereas, at least when the two-point correlation of the velocity decays fast enough in space, B_2 is, except by a factor, approximately the angular momentum. Anyway we will assume (6) to hold in the integral range, aware that it is at best an approximation.

B_2 would also be constant if the so-called Loitsianski integral exists, a point defended in [7]. Most authors prefer a weak dependence of B_2 in time, $B_2(t) \sim t^\gamma$, $\gamma \sim 0.25$ [8]. For large times, most of the energy is assumed to be contained in the largest scales where this description of E holds, so the size of B_2 is crucial. The above formula is assumed to hold in the so-called integral range. The hypothesis that energy cascades without viscous loss in the next range prompted Kolmogorov (see e.g. [9]) to propose a potential decay of $E(k)$ in this range, called the inertial one: details will be presented later. This picture was amended in dimension two by Kraichnan [10, 11]. The inertial range is followed by the dissipative one, where E decreases exponentially and therefore its contribution to the total energy is minimal. The arguments are debatable, but the success of this description is undeniable: experiments, observations and models have repeatedly confirmed the essential correctness of the scheme. They also show a rather good agreement in the transition from the integral to the inertial ranges: a certain smoothing takes place, specially in dimension three, but the point of contact k_i between the two ranges is located very approximately at the intersection of the graphs of the respective functions of k [12].

Our main purpose is to take advantage of the above description of the spectrum to integrate (3) exactly, and therefore to find the energy decay rate irrespective of any asymptotics.

It may be argued that several of the parameters involved in the spectrum structure are imperfectly known, and therefore the results will depend excessively on speculative values. In particular, we mention the behavior at the junction of the integral and inertial spectra, and the corrections to the Kolmogorov-Kraichnan exponents due to finite viscosity. However, the fact that both energy and enstrophy are integrals means that their dependence with respect to the function to be integrated is very weak: small changes in the function at small portions of the range will have an even smaller response in the integral. Thus our results are robust with respect to parameter variation. Second, one of our objectives is to provide a criterion to assert the compatibility of classical spectra with the observed flow evolution. While the energy depends on a single integral and therefore may be rather easily found both in theoretical and experimental models, the spectrum is more costly and subject to errors. Thus we know that a departure from predicted energy decay rates will mean a deviation from the classical spectrum, without the need to find it explicitly. This method may conceivably be applied to other turbulent phenomena for which numerical integration of the system has not been performed as often as for standard hydrodynamics. Examples of this are magnetohydrodynamics (MHD) and electron magnetohydrodynamics (EMHD). The mechanism of energy transfer, and therefore the shape of the inertial range, is still a much debated topic in MHD: whether it follows Kolmogorov or Iroshnikov-Kraichnan statistics is a point of contention (see e.g. [13, 14, 15, 16]). As for EMHD, some information of the spectrum is available [17]. A comparison of (3) with the measured energy decay rate may therefore provide information on the shape and extension of the inertial range in these cases.

2 Three-dimensional homogeneous turbulence

According to the Kolmogorov phenomenology, in the inertial range

$$E(k,t) = C\varepsilon(t)^{2/3}k^{-5/3}, \tag{7}$$

where $\varepsilon(t)$ represents the energy flux from larger to smaller scales: we will comment later on its traditional meaning. For the moment, let us take it simply as an unknown function, and let us define $v(t) = C\varepsilon(t)^{2/3}$ to simplify the equations. C is the Kolmogorov-Obukhov constant: although it should be universal, its value varies somewhat with experiments, $C \simeq 1.4 - 2$.

The disipative cut-off is found by equating in the Navier-Stokes equation the dissipative and advective terms. Its theoretical value is

$$k_d(t) = \varepsilon(t)^{1/4}\nu^{-3/4} = C^{-3/8}\nu^{-3/4}v(t)^{3/8}. \tag{8}$$

However the experimental value of k_d may vary by as much as a factor of ten from the theoretical one [9], so we will allow a multiplicative nondimensional constant A in its definition, and by calling $\alpha = AC^{-3/8}$, we get

$$k_d(t) = \alpha\nu^{-3/4}v(t)^{3/8}. \tag{9}$$

As stated before, we will ignore the contribution of the dissipative range to both the energy and the enstrophy: the exponential decay of the spectrum there makes its contribution extremely small.

We will consider the zero mean case and therefore we assume that in the integral range E has the form

$$E(k,t) = B_2(t)k^4. \tag{10}$$

Therefore the integral cut-off is determined by the continuity of the spectrum, which is a well attested fact:

$$B_2(t)k_i(t)^4 = v(t)k_i(t)^{-5/3}. \tag{11}$$

The smoothing present in the graph is limited to a small neighborhood of k_i [18] and therefore its effect upon the total energy may be ignored without serious error. Notice that since $v(t)$ will decrease in time and $B_2(t)$, if anything, grows with t, k_i should decrease in time, a point confirmed in modelization. Adding the contribution of both ranges to the total energy, we get

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^{k_i(t)} B_2(t)k^4\,dk + \frac{1}{2}\int_{k_i(t)}^{k_d(t)} v(t)k^{-5/3}\,dk \\ &= \frac{17}{20}B_2(t)^{2/17}v(t)^{15/17} - \frac{3}{4}\alpha^{-2/3}\nu^{1/2}v(t)^{3/4}. \end{aligned} \tag{12}$$

Notice that the last negative term does not represent the contribution of the second integral, which is of course positive.

As for the enstrophy, since $\Omega(k,t) = 2k^2E(k,t)$,

$$\begin{aligned} \Omega(t) &= \int_0^{k_i(t)} B_2(t)k^6\,dk + \int_{k_i(t)}^{k_d(t)} v(t)k^{1/3}\,dk \\ &= -\frac{17}{28}B_2(t)^{-4/17}v(t)^{21/17} + \frac{3}{4}\alpha^{4/3}\nu^{-1}v(t)^{3/2}. \end{aligned} \tag{13}$$

According to the Kolmogorov phenomenology, ε represents the energy dissipation rate $\varepsilon = \nu\Omega$. From a strict mathematical viewpoint, this overdetermines the problem and makes unlikely the existence of a solution to (3) with the expressions (12) and (13). In fact, rewriting Ω as a function of ε we obtain

$$\Omega(t) = -\frac{17}{28}B_2(t)^{-4/17}C^{21/17}\varepsilon(t)^{14/17} + \frac{3}{4}\alpha^{4/3}C^{3/2}\nu^{-1}\varepsilon(t),$$

which is not exactly $\nu^{-1}\varepsilon$. However, we must remember that the Kolmogorov theory is an asymptotic one, holding for very small ν (although as asserted, the spectrum shape is valid for a rather large range of viscosities). For ν very small, the second term in the previous formula predominates, and this has the form $\nu^{-1}\varepsilon$ times an universal constant, introduced to account for different experimental results, and which should be one in the Kolmogorov framework. The agreement is satisfactory, but we must remember that we use only the spectrum shape as an hypothesis, not the specific traditional meaning of some variables.

Since the energy must be positive, the function v must lie within a certain range: specifically, it must occur

$$\frac{17}{15}B_2(t)^{2/17}v(t)^{9/68} > \alpha^{-2/3}\nu^{1/2}, \tag{14}$$

which given the small value of the viscosity is satisfied for very small v (and therefore very small energy). A negative value of the energy would mean $k_i > k_d$, which is absurd. Since the enstrophy should also be positive, we must have

$$\frac{21}{17}B_2(t)^{4/17}v(t)^{9/34} > \alpha^{-4/3}\nu. \tag{15}$$

This condition implies that the energy is positive, since $(17/15)^2 > 21/17$. This must be taken into account in the integration of (3). The initial value $v(0)$ should be taken so that $\Omega(0) > 0$, and integration can be carried out for as long as $\Omega(t) > 0$. If for a certain t_0 we reach a value of v such that (15) becomes an equality and $\Omega(t_0) = 0$, since $E(t_0) > 0$ and $E' = -\nu\Omega$, E does not decrease beyond this value, so that $E(t) \geq E(t_0)$. The fact that the energy has a positive lower value is of course unphysical in the long run, which emphasizes the fact that our description of the spectrum is transitory. It is true that this minimum value of the energy decreases with ν, but we cannot expect that integration of (3) for fixed viscosity will yield a decrease of E like a negative power of t: instead we will find $E(t) \sim E_0 + bt^{-r}$, where E_0 is the minimum of energy. Two parameters are of interest: this minimal energy E_0 compatible with our power spectrum, and the rate r with which E decreases to E_0. E_0 may be found a priori in certain cases: when B_2 does not depend on t, the equation has two critical points, 0 and the value $v_0 > 0$ for which the expression of Ω vanishes. In our case this is

$$v_0 = \left(\frac{17}{21}\right)^{34/9} B_2^{-8/9}\alpha^{-136/27}\nu^{34/9}. \tag{16}$$

Since v_0 is an attractive node and we start with $v(0)$ such that $\Omega(0) > 0$, any solution will tend to E_0 given by (12) with $v = v_0$, i.e.

$$\begin{aligned} E_0 &= \left(\frac{17}{20}\left(\frac{17}{21}\right)^{10/3} - \frac{3}{4}\left(\frac{17}{21}\right)^{17/6}\right)\alpha^{-40/9}\nu^{10/3}B_2^{-2/3} \\ &\simeq (0.08119)\,\alpha^{-40/9}\nu^{10/3}B_2^{-2/3}. \end{aligned} \tag{17}$$

Anyway, both E_0 and r will be computed numerically: E_0 by studying the limit of $E(t)$ for large t. r may be found in several different ways: the simplest is to consider

$$r = -\lim_{t\to\infty} \frac{\log(E(t) - E_0)}{\log t}. \tag{18}$$

While this works reasonably well, it needs to know beforehand the value E_0. Also, for $E(t)$ near E_0, rounding errors may provide a negative $E(t)-E_0$, which may lead to problems with the logarithm. The next method has turned out to be far more robust: take any fixed $c > 1$, and let

$$W(t) = \frac{E(tc^2) - E(tc)}{E(tc) - E(t)}. \tag{19}$$

If E tends to a constant like a negative power of t, $W(t)$ will stay away from zero. The power t^{-r} may be found by

$$r = -\frac{1}{\log c} \lim_{t\to\infty} \log(W(t)). \tag{20}$$

A moment's reflection will show us that for an expression of $E(t) \sim a + bt^{-r}$, r is indeed this limit. The same arguments apply to the two-dimensional case.

Classically it is assumed that $\nu \to 0$, so that $E_0 \to 0$ and $k_i \ll k_d$. In this situation, we can obtain some analytic estimates: The dominant terms in energy and enstrophy are respectively

$$\begin{aligned} E(t) &\sim \frac{17}{20} B_2(t)^{2/17} v(t)^{15/17}, \\ \Omega(t) &\sim \frac{3}{4} \alpha^{4/3} \nu^{-1} v(t)^{3/2}. \end{aligned} \tag{21}$$

The resulting equation yields, except for numerical factors (independent of ν) the solution

$$E(t) \sim \left(E(0)^{-7/10} + \int_0^t B_2(s)^{-1/5}\, ds \right)^{-10/7}, \tag{22}$$

so that, for a behavior of $B_2(t) \sim t^\gamma$, we have

$$E(t) \sim t^{(-10/7)+(2\gamma/7)}, \tag{23}$$

as foreseen by the classical theory. However, this approximation essentially means that all the energy is concentrated in the vicinity of k_i and all the enstrophy in the vicinity of k_d, which is rather rough. Therefore, we have integrated numerically (3) with the values of energy and enstrophy given respectively by (12) and (13). Not only the possible value of the limit r is relevant: the value of (20) for large but finite time is perhaps even more meaningful, specially if for a long time the rate of decay follows a well defined law. We have already emphasized that the spectrum does not maintain the same form for ever. In fact, while the decay of the energy to a threshold level is robust in all the cases studied, we have found that r tends to stabilize for a large interval of time, after which its behavior tends to be more erratic. It is unlikely that this effect is only due to numerical errors: apparently there exist other terms in the energy evolution besides $E(t) \sim a + bt^{-r}$ which eventually become dominant. The interval where the exponent r behaves well is approximately proportional to the size of B_2. In all the cases it is ample enough to account for the evolution of r for as much time as one could expect the spectrum structure to hold.

The following graphics show the evolution of the energy and minus the exponent of convergence r for $B_2(t) = 1$, 10, 100 and 1000. We have taken a logarithmic scale on the time axis, although the numbers indicate the correct value of t, and the trial values $\alpha = 1$, $\nu = 1/16$.

The initial values are unimportant and are chosen only to make a clear picture: the rate of decay depends only on the size of B_2.

INSERT FIGURE 1 HERE

The value of $-r$ is plotted in Figure 2. We see that near the end of the scale the value of $-r$ for $B_2 = 1$ begins the irregular behavior we mentioned earlier. Again, the limit values of $-r$ do not depend on the initial conditions. They lie in a band between 1.4 and 1.7.

INSERT FIGURE 2 HERE

The same graphs are shown in Figure 3 for the energy and the exponent for another standard case: $B_2 = \beta t^{0.25}$ for $\beta = 1$, 10, 100 and 1000. The graphs are practically identical to the previous ones.

INSERT FIGURE 3 HERE

By contrast, we see in Figure 4 that the convergence of $-r$ is better for this case: the limit seems to lie between 1.3 and 1.6.

INSERT FIGURE 4 HERE

Notice that for much of its evolution, the curves are nearly equidistant. This suggests a behavior of the exponent as a function of β, both in the constant and in the time-dependent cases, as

$$-r(\beta) \sim -r(1) - k(t) \log \beta, \tag{24}$$

where $k(t)$ tends to a constant. By integrating further, one in fact finds $k(t) \to 0$, so that the limit exponent is the same for all cases. However, we refrain from stating formally this by two reasons: the erratic behavior of $-r$ for t very large and β small, and our repeated warnings about the duration of the spectrum structure.

3 Two-dimensional homogeneous turbulence

Kraichnan [10] proposed a direct cascade of enstrophy instead of energy for two-dimensional turbulence. A scaling argument shows that the energy in the inertial range decays like

$$E(k,t) = \eta(t)^{2/3} k^{-3} \tag{25}$$

where η represents now the enstrophy flux. The exponent -3 is not as universal as $-5/3$ in three-dimensional turbulence: logarithmic corrections of the type

$$E(k,t) = \eta(t)^{2/3} k^{-3} \left(a + \log\left(\frac{k}{k_i}\right)\right)^{-1/3} \tag{26}$$

have been proposed [19], and experiments show often a decay in $k^{-\mu}$, $\mu > 3$ (see e.g. [20]). Other models, however, show reasonable concordance with k^{-3} and we will take this for concretion: the method may be applied to any other decrease. Near $k = 0$, the Taylor expansion of E starts as mentioned previously by

$$E(k,t) = B_0 k + B_2(t) k^3 + o(k^3). \tag{27}$$

We will assume that the flow has mean zero and therefore $B_0 = 0$. Modelization shows that the shape of the integral spectrum follows quite accurately the graph of k^3 and that the transition from the integral to the inertial range resembles strongly the union of the graphs of k^3 and k^{-3} [21]. Thus, denoting for simplicity $v(t) = \eta(t)^{2/3}$, the junction point should satisfy

$$B_2(t) k_i(t)^3 = v(t) k_i(t)^{-3}, \tag{28}$$

and therefore

$$k_i(t) = B_2(t)^{-1/6} v(t)^{1/6}. \tag{29}$$

On the other hand, the dissipative cut-off lies at

$$k_d(t) = \nu^{-1/2} \eta(t)^{1/6} = \nu^{-1/2} v(t)^{1/4}. \tag{30}$$

For simplicity, and without modifying the rate of decay of the energy, we will omit possible constants in front of k_d. Thus the energy becomes

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^{k_i(t)} B_2(t) k^3 \, dk + \frac{1}{2}\int_{k_i(t)}^{k_d(t)} v(t) k^{-3} \, dk \\ &= \frac{3}{8} B_2(t)^{1/3} v(t)^{2/3} - \frac{1}{4}\nu v(t)^{1/2}, \end{aligned} \tag{31}$$

whereas the enstrophy is

$$\begin{aligned}\Omega(t) &= \int_0^{k_i(t)} B_2(t)k^6\,dk + \int_{k_i(t)}^{k_d(t)} v(t)k^{-1}\,dk \\ &= \frac{1}{6}B_2(t)k_i(t)^6 + v(t)\log\left(\frac{k_d(t)}{k_i(t)}\right) \\ &= \frac{1}{6}v(t) + v(t)\log\left(\nu^{-1/2}B_2(t)^{1/6}v(t)^{1/12}\right). \end{aligned} \tag{32}$$

As in the three-dimensional case, v cannot be arbitrarily small for the energy to be positive. In this instance, the inequality

$$\frac{3}{2}B_2(t)^{1/3}v(t)^{1/6} > \nu, \tag{33}$$

must hold. As before, this occurs if the enstrophy is positive:

$$e^{1/3}B_2(t)^{1/3}v(t)^{1/6} > \nu, \tag{34}$$

since $(3/2)^3 > e$.

We can study as well the case where B_2 is constant in time, although this is less likely in dimension two. If this occurs, the critical points of the equation are the two zeroes of the enstrophy, i.e. 0 and

$$v = e^{-2}\nu^6 B_2^{-2}, \tag{35}$$

so that the minimum value of the energy for this spectrum to hold is

$$E_0 = \left(\frac{3}{8}e^{-4/3} - \frac{1}{4}e^{-1}\right)\nu^4 B_2^{-1} \simeq (0.00688)\,\nu^4 B_2^{-1}. \tag{36}$$

The approximation performed in the three-dimensional case should now be less succesful, since the quicker decrease in the inertial range means that while $k_d > k_i$, it is not much larger. Indeed, simulations show in many cases $k_d/k_i < 100$ [21]. It is therefore acceptable to take $\log(k_d/k_i)$ as a magnitude of order 1. Thus, if we suppress the term k_d^{-2} in the energy and take the enstrophy with the order of v, the approximation of (3) becomes, omitting numerical constants

$$\frac{d}{dt}(B_2(t)^{1/3}v(t)^{2/3}) \sim -v(t), \tag{37}$$

whose solution yields

$$E(t) \sim \left(E(0)^{-1/2} + \int_0^t B_2(s)^{-1/2}\,ds\right)^{-2}, \tag{38}$$

so that, if $B_2(s) \sim s^\beta$, $E(t) \sim t^{\beta-2}$. For $\beta = 1$, the energy decays as t^{-1}, as observed in [21] for this particular election of Reynolds number: the enstrophy decays like t^{-2}, as predicted by Batchelor [11]. However, other rates have been observed, although for $\beta > 2$ the energy would actually increase, which is unacceptable.

As in the three-dimensional case, we have integrated numerically the evolution of energy and exponent for the classical cases $B_2 = \beta$ and $B_2 = \beta t$, $\beta = 1$, 10, 100 and 1000. Figure 5 shows the values of the energy for B_2 constant in time. After a slow initial decay, the energy tends rapidly to the threshold value.

INSERT FIGURE 5 HERE

The evolution plotted in Figure 6 shows that the convergence of the exponent to a common value (about 1.75) is much more obvious in this two-dimensional case. Notice, however, that the empirical law (24) still holds: the graphics tend to be equidistant. A different choosing of initial conditions could mask this effect in the short run, but it establishes itself rather quickly.

INSERT FIGURE 6 HERE

Figure 7 shows that the energy decays more slowly when $B_2 = \beta t$. This is to be expected, since a growth of B_2 with time means an increase of energy in the largest scales. Also, for larger β, the rate of decay is lower.

INSERT FIGURE 7 HERE

Finally, Figure 8 indicates convergence of $-r$ to a value near 1. We have needed to stretch the scale of time up to 10^5 because of the slower rate of convergence in this case; in the two-dimensional case the problems controlling $-r$ are less serious than in the three-dimensional one. Again, (24) holds satisfactorily.

INSERT FIGURE 8 HERE

4 Conclusions

The classical description (Kolmogorov-Kraichnan) of the power spectrum in homogeneous turbulent flows entails a functional relation between energy and enstrophy which may be exploited to find the energy evolution without the heavy computational cost of numerical integration of the whole Navier-Stokes system. Integration of the ordinary differential equation of energy decay will provide us with estimates appropriate for the intermediate stages of time and small, but not vanishingly small, viscosity which are in fact the most realistic and to whom classical asymptotics arguments do not apply. This is studied here both in dimensions two and three, allowing for several forms of the Taylor coefficient $B_2(t)$ of the energy proposed by different authors. In contrast to what happens in the asymptotic limits $t \to \infty$ and $\nu \to 0$, energy will have a positive minimum here, which is perfectly reasonable since the standard shape of the power spectrum is transitory and our results only apply to this stage. These minima as well as the rate of decay of the energy towards them are studied theoretically and numerically. The decay of the energy towards this minimum is a robust feature in all the cases, but the exponent of decay depends strongly on the number of dimensions and the time dependence of B_2. In the three-dimensional case these exponents lie in a narrow band: separation between them seems to be proportional to $\log \beta$, where $B_2(t) = \beta t^a$ for the classical values $a = 0$, $a = 0.25$. They appear to converge to a common value in dimension two, this time for the standard values $a = 0$ and $a = 1$; again a logarithmic separation among the graphs is noticeable. Since the total energy is much less costly to calculate than the full power spectrum, these results may be used to find if any given turbulent flow follows or not Kolmogorov-Kraichnan statistics, without the need to compute the Fourier components of the velocity.

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FIGURE CAPTIONS

1 (nunez fig01.eps): FIG.1. Evolution of the energy for time-independent B_2.

2 (nunez fig02.eps): FIG.2. Exponent of convergence for time-independent B_2.

3 (nunez fig03.eps): FIG.3. Evolution of the energy for $B_2 = \beta t^{0.25}$.

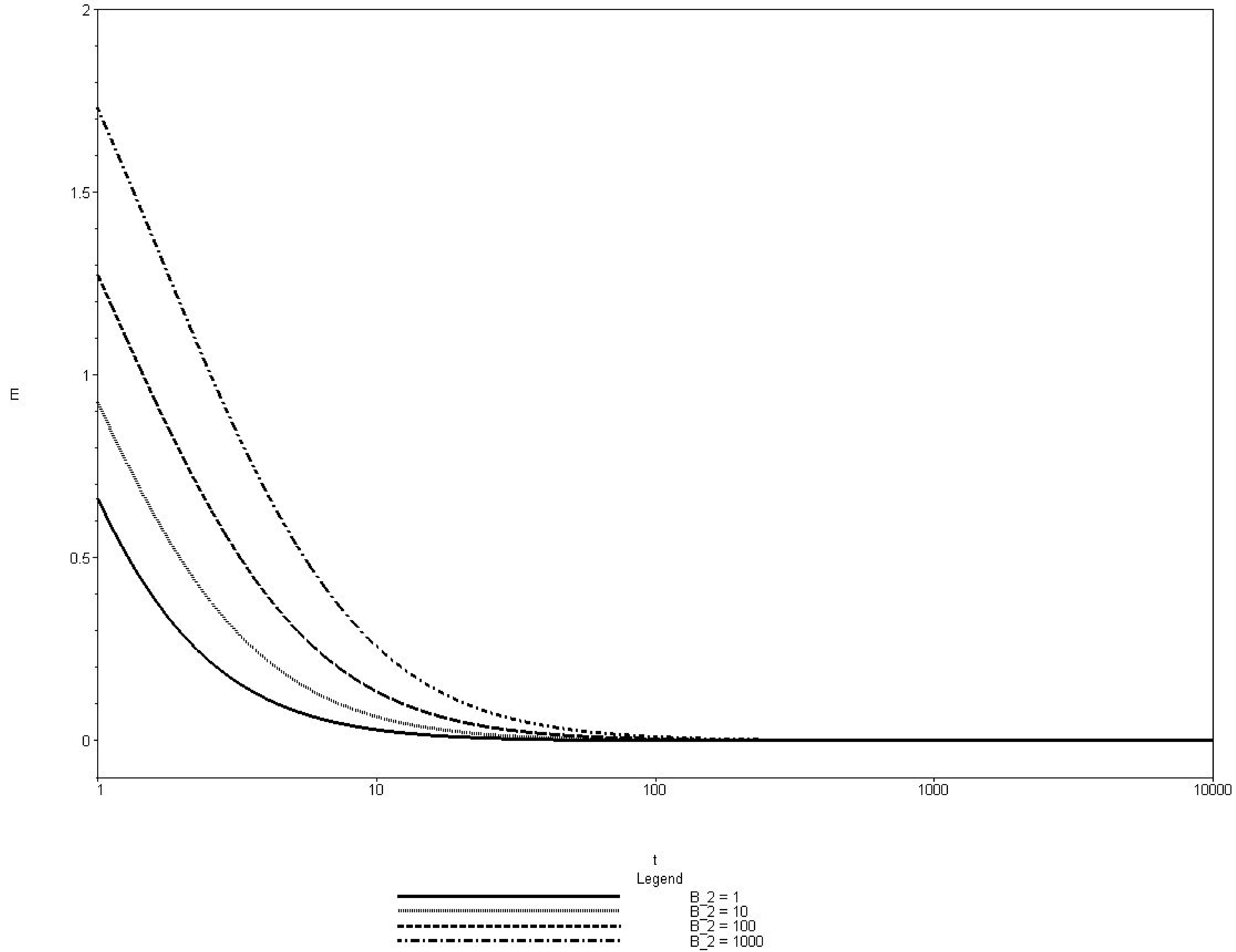
4 (nunez fig04.eps): FIG.4. Exponent of convergence for $B_2 = \beta t^{0.25}$.

5 (nunez fig05.eps): FIG.5. Evolution of the energy for time-independent B_2.

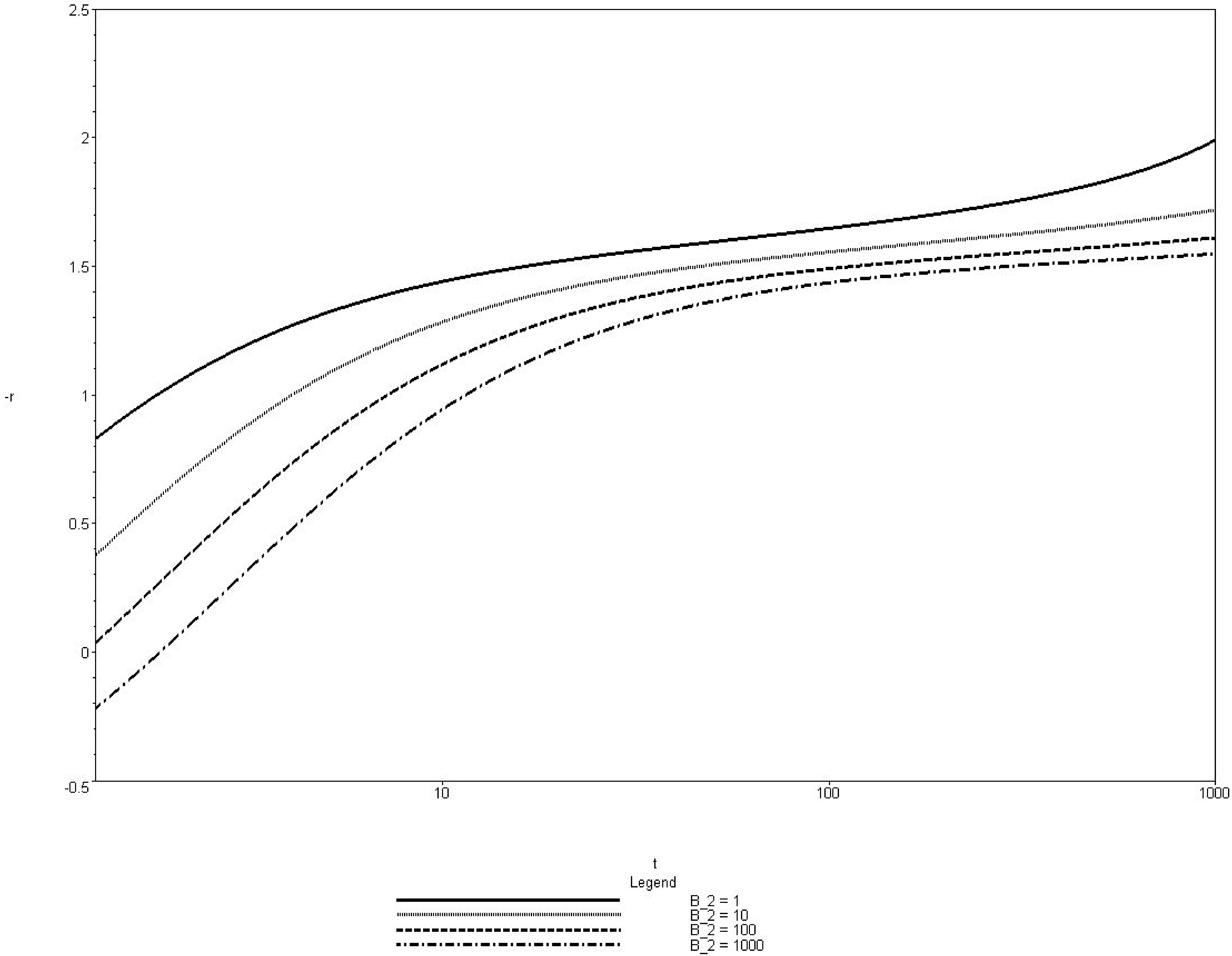
6 (nunez fig06.eps): FIG.6. Exponent of convergence for time-independent B_2.

7 (nunez fig07.eps): FIG.7. Evolution of the energy for $B_2 = \beta t$.

8 (nunez fig08.eps): FIG.8. Exponent of convergence for $B_2 = \beta t$.



2
1.5
1
0.5
0
E
1
10
100
1000
10000
t
Legend
B_2 = 1
B_2 = 10
B_2 = 100
B_2 = 1000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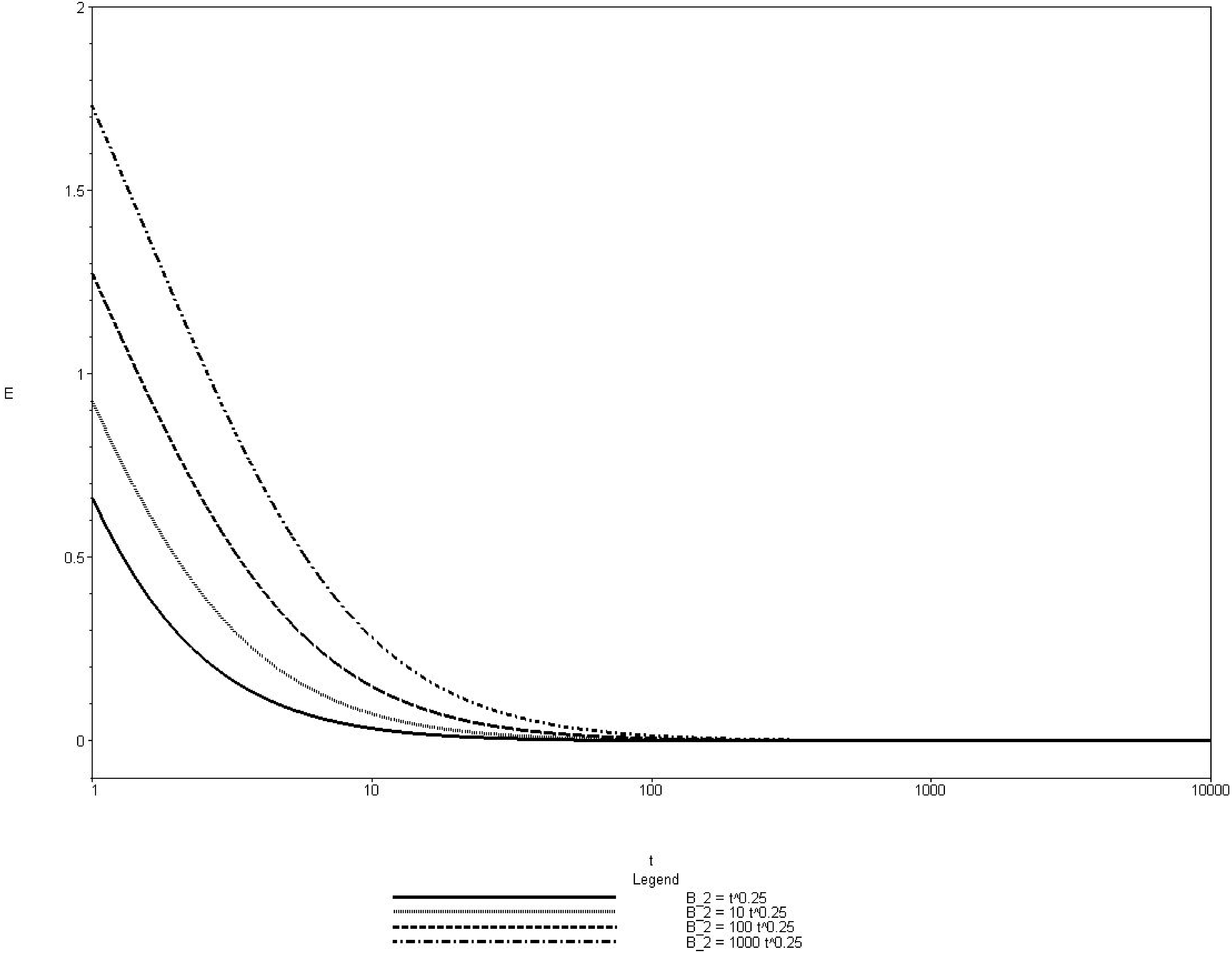
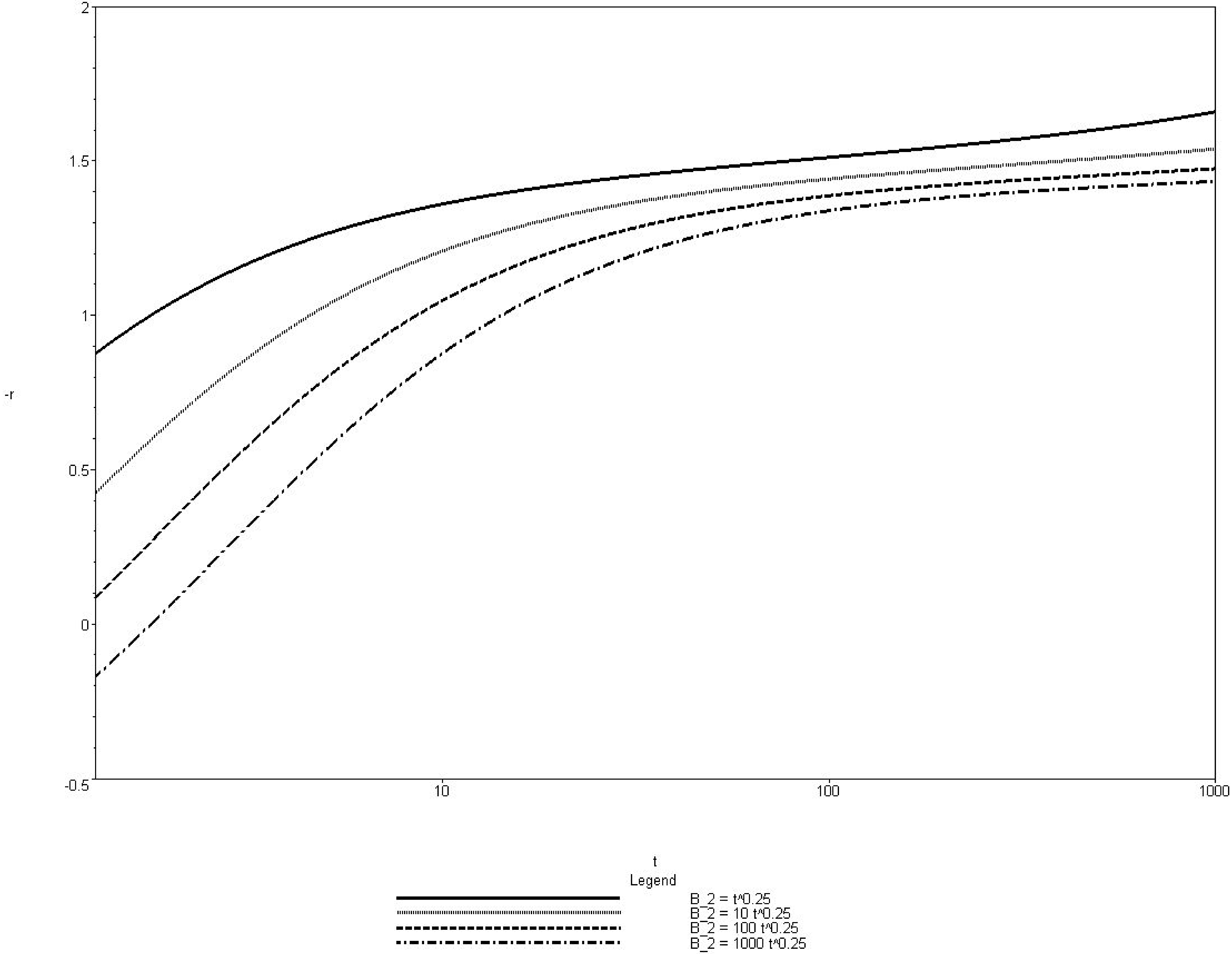
2
1.5
1
0.5
0
-0.5
-r
10
100
1000
t
Legend
B_2 = 1
B_2 = 10
B_2 = 100
B_2 = 1000


2
1.5
1
0.5
0
E
1
10
100
1000
10000
t
Legend
B_2 = t^0.25
B_2 = 10 t^0.25
B_2 = 100 t^0.25
B_2 = 1000 t^0.25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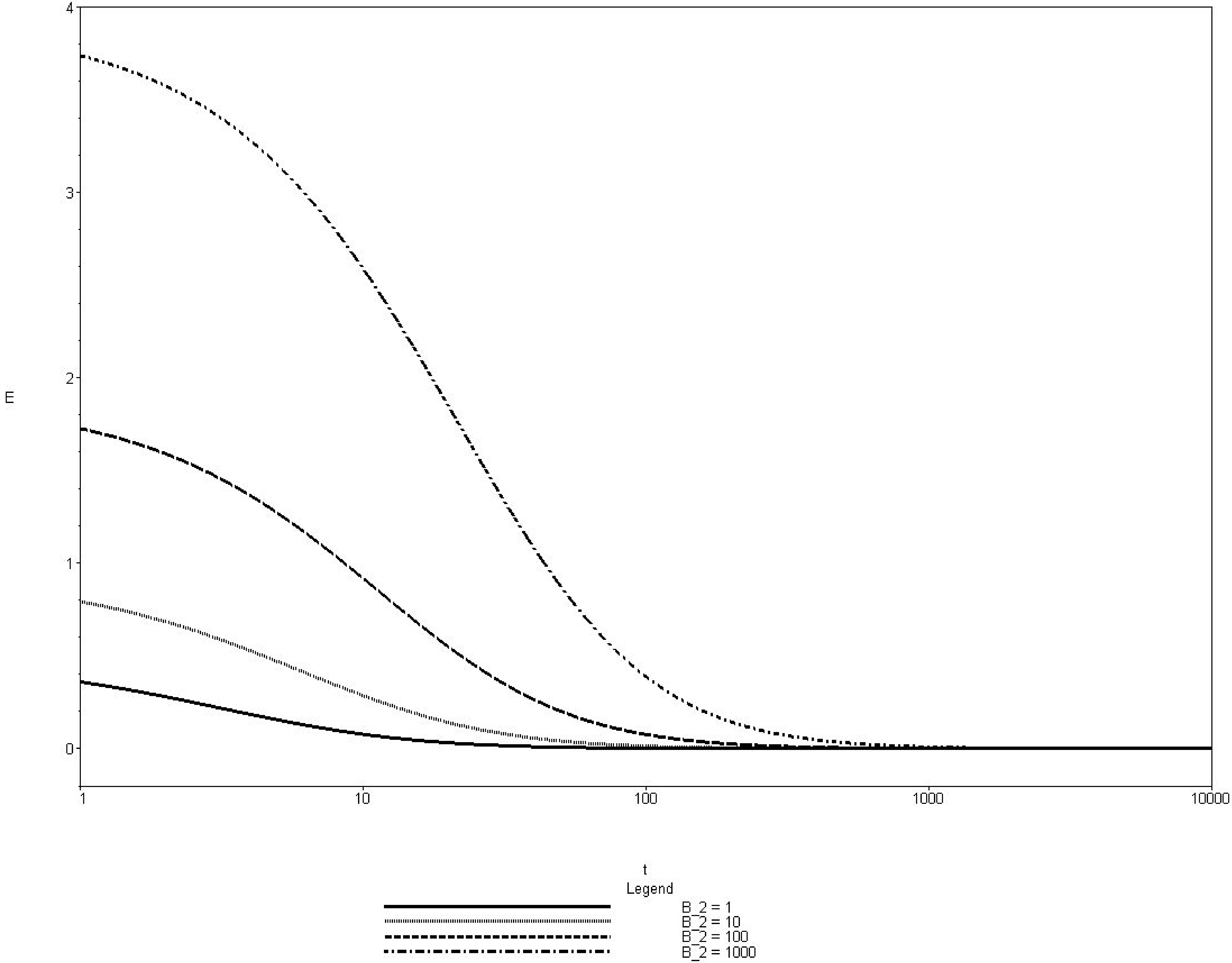
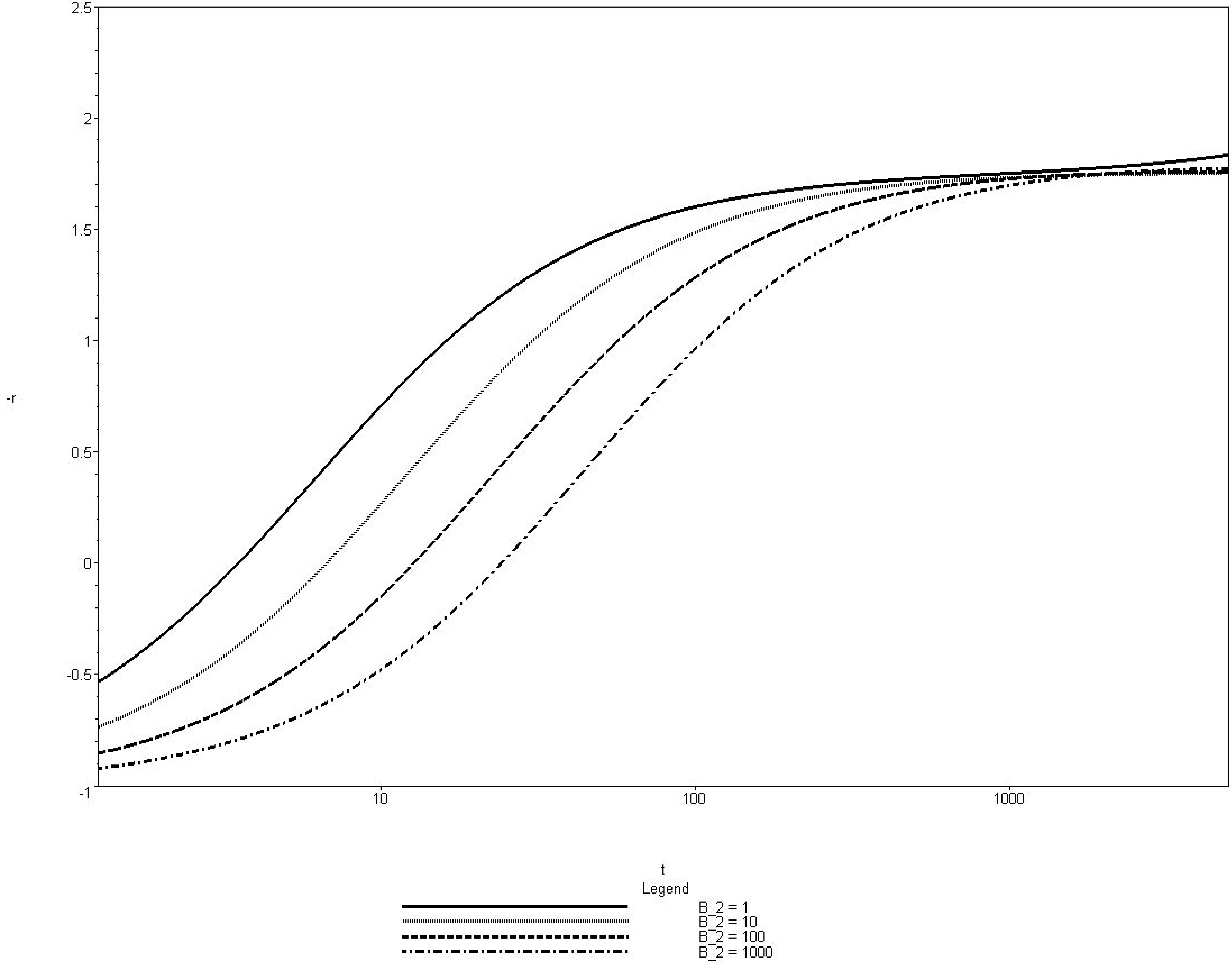
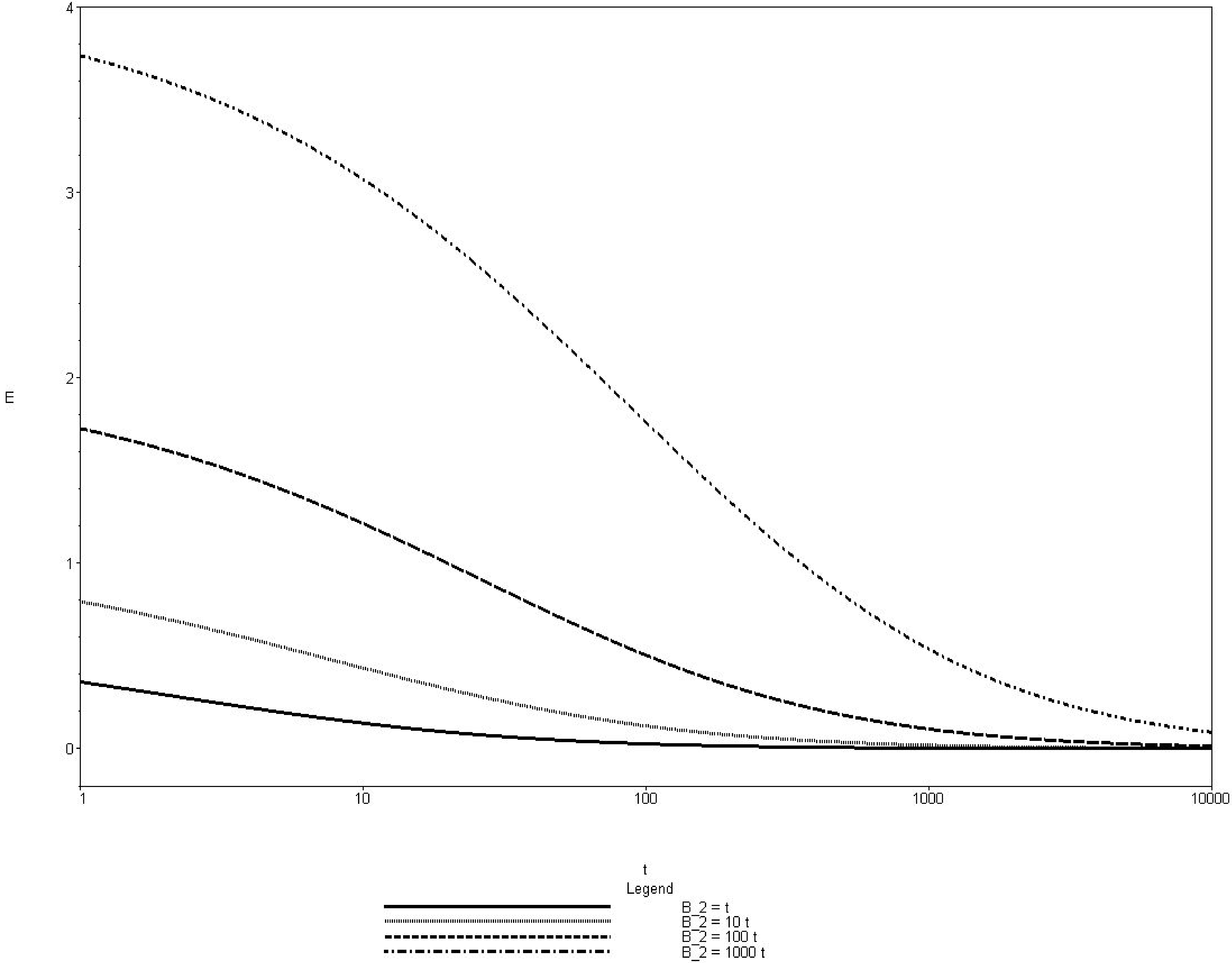
1.5
1
0.5
0
-0.5
-r
10
100
1000
t
Legend
B_2 = t^0.25
B_2 = 10 t^0.25
B_2 = 100 t^0.25
B_2 = 1000 t^0.25

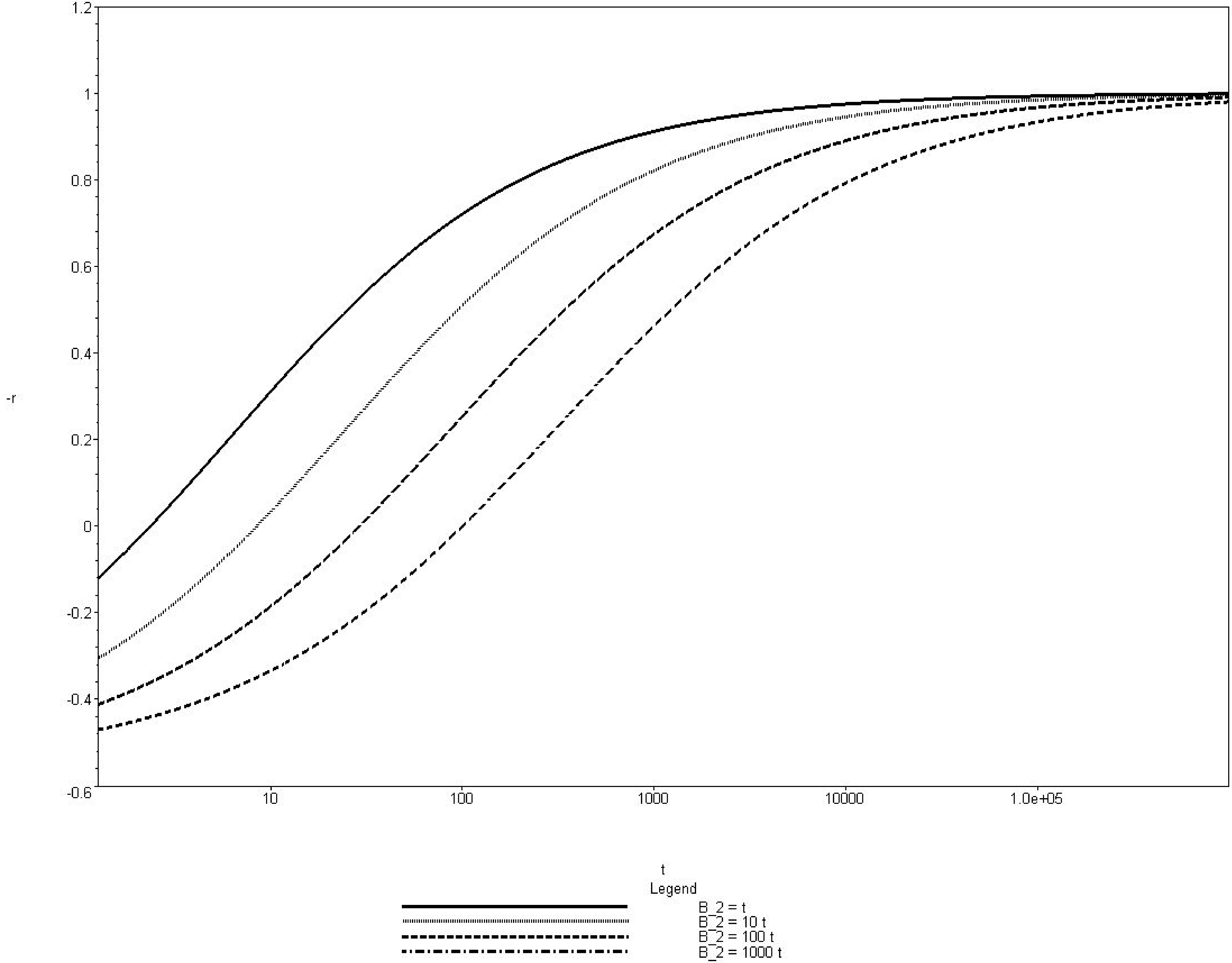


4
3
2
1
0
E
1
10
100
1000
10000
t
Legend
B_2 = 1
B_2 = 10
B_2 = 100
B_2 = 1000


2.5
2
1.5
1
0.5
0
-0.5
-1
-r
10
100
1000
t
Legend
B_2 = 1
B_2 = 10
B_2 = 100
B_2 = 1000



4
3
2
1
0
E
1
10
100
1000
10000
t
Legend
B_2 = t
B_2 = 10 t
B_2 = 100 t
B_2 = 1000 t


1.2
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0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-r
10
100
1000
10000
1.0e+05
t
Legend
B_2 = t
B_2 = 10 t
B_2 = 100 t
B_2 = 1000 t