# *Constructions of 1-perfect partitions on the $n$-cube $(\mathbb{Z}/2)^n$

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## Extended Abstract

Let $\mathbb{F}^n$ be a vector space of dimension $n$ over $\mathbb{Z}/2$. A *binary 1-perfect code* $C$ of length $n$ is a subset of $\mathbb{F}^n$, such that every $x \in \mathbb{F}^n$ within distance 1 from exactly one codeword of $C$. If we consider distance $r \neq 1$ instead of 1, we have trivial codes or the binary Golay code, so we will henceforth use the word "perfect" to refer specifically to 1-perfect codes.

The length $n$ of a perfect code is $n = 2^m - 1$ for some $m \geq 3$. The linear perfect codes exist $\forall m \geq 3$ and are unique up to isomorphism (they are the well-known Hamming codes).

A *1-perfect partition* is a partition of the space $\mathbb{F}^n$ into $n+1$ perfect codes $C_0, C_1, \ldots, C_n$. We can assume the zero vector is in $C_0$ and the vectors having a one in the *ith* coordinate and zeroes elsewhere, $e_i$, are in $C_i$, $\forall i \in \{1, \ldots, n\}$. Given a perfect code $C$ of length $n = 2^m - 1$ we know that there always exists $n+1$ translates of $C$, $C+e_0$, $C+e_1, \ldots,$ $C+e_n$, that form a 1-perfect partition of $\mathbb{F}^n$, we will call this the trivial partition.

Two partitions $C_0, C_1, \ldots, C_n$ and $D_0, D_1, \ldots, D_n$ are *isomorphic* if there exists a permutation $\pi$ of the coordinates which maps the vectors of each class into the vectors of a class in the second partition, that is $\forall j \in \{1, \ldots, n\}$ $D_j = \{\pi(C_i)\}$ for some $i \in \{1, \ldots, n\}$. Two partitions $C_0, C_1, \ldots, C_n$ and $D_0, D_1, \ldots, D_n$ are *equivalent* if there exists a permutation $\pi$ of the coordinates and a translation $\tau$ such that for all classes $D_j$ there exists a class $C_i$ such that $D_j = \{\pi(C_i) + \tau\}$.

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We will see two ways of constructing 1-perfect partitions on the $n$-cube $(\mathbb{Z}/2)^n$. The first one is by using a generalized Slov'eva-Phelps' switching technique [4]-[5]. We will start not only from trivial partitions, but from any partition $C*e_0,\ C*e_1,\ldots,\ C*e_n$, where $C*e_0=C$ and for all $x\in C,\ x*e_i$ is the only vector in class $C*e_i$ at distance one apart from $x$ (if $x*e_i=x+e_i$, we have the trivial partition). This technique consist in construct two new 1-perfect codes and then construct another, different partition of $\mathbb{F}^n$ with some restrictions.

First of all we take two classes $C$ and $C*e_i$ of the partition and define a graph $(V,E)$ in $C$ by $V=C$

$$(x,y)\in E\Leftrightarrow\begin{cases} d(x,y)=3\\ d(x,y*e_i)=2\ or\ d(y,x*e_i)=2\end{cases}$$

Let $S\subset C$ be a connected component in the above graph, and let $S'\subset C*e_i$, be the subset in $C*e_i$ defined by $S'=\{x*e_i,\quad\forall x\in S\}$. If we switch $S$ and $S'$, we define new classes $D=(C\backslash S)\cup S'$ and $H=D*e_i$ that will be perfect codes.

The initial partition $C,\ C*e_1,\ldots,\ C*e_n$, and the partition $D,\ C*e_1,\ldots,\ C*e_{i-1},\ D*e_i,\ C*e_{i+1},\ldots,\ C*e_n$ will be different if the graph $(V,E)$ has more than one connected components for some $i\in\{1,\ldots,n\}$, because the new classes $D$ and $H$ are different from $C$ and $C*e_i$.

It is not always possible to obtain a different partition with this construction. The problem is that there exist partitions of $\mathbb{F}^n$ such that $\forall i\in\{1,\ldots,n\}$ the graph has only one connected component.

The second way of constructing 1-perfect partitions on the $n$-cube $(\mathbb{Z}/2)^n$ is by using algebraic distance-compatible structures defined on $(\mathbb{Z}/2)^n$ such as in [3].

A binary operation on $(\mathbb{Z}/2)^n$ is distance-compatible if for all $v\in(\mathbb{Z}/2)^n$ and a unitary vector $e_i$, we have $d(v,v*e_i)=1$ and for all $e_i$, the induced map $v\rightarrow v*e_i$ is one-to-one.

There are too many distance-compatible operations in $(\mathbb{Z}/2)^n$, so it is important to focus our attention to the binary distance-compatible operations with fulfills more conditions, for example we can take quasigroup structures or group (abelian or not abelian) structures and so on.

We prove in this paper that in some cases if we take a distance-compatible operation such that for all $v,w,s\in F^n$, $d(v*s,w*s)=d(v,w)$, then the operation is a quasigroup structure of exponent 2 or 4.

We know some elementary cases in which $(\mathbb{Z}/2)^n$ has an algebraic structure of exponent 2 or 4, for example when we consider $(\mathbb{Z}/2)^n$ as a direct product of $\mathbb{Z}/2,\mathbb{Z}/4$ or $\mathbb{Q}_8$ groups (this last structure means the quaternion group of eight elements).

By using this last consideration in the particular case $n = 7$ it is possible to see $(\mathbb{Z}/2)^7$ as $\mathbb{F}^7$, $\mathbb{F}^5 \oplus \mathbb{Z}/4$, $\mathbb{F}^3 \oplus (\mathbb{Z}/4)^2$, $\mathbb{F}^3 \oplus \mathbb{Q}_8$ or $\mathbb{F} \oplus \mathbb{Z}/4 \oplus \mathbb{Q}_8$. Each one of these possibilities gives us a non-isomorphic 1-perfect partition in $(\mathbb{Z}/2)^7$.

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